Mini Putnam Exam II

These are all taken from Putnam Exams given from 1986–1988. Give yourself 3 hours and write up your solutions as best you can. Submit, and I'll constructively criticize your writeup. The idea is NOT to look up solutions.

I. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

II. What is the units (i.e., the rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$.

III. Evaluate $\displaystyle\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where Arccot t for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.

IV. Evaluate $\displaystyle\int_2^4 \frac{\sqrt{\ln(9-x)}dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$.

V. Let R be the region consisting of the points x, y of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

VI. Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.