Mini Putnam Exam I

These are all taken from prevvious Putnam Exams.

1. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n,\ m = 0,\ 1,\ 2,\ \ldots)$? Justify your answer.

2 Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

3 Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for every pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y) \\ g(x+y) &= f(x)g(y) + g(x)f(y) \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

4 Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is a member of the set $\{0,\ 1,\ 4,\ 9,\ 16, \ldots\}$. a is **within** n of b if $b - n \leq a \leq b + n$.)

5 What is the units (i.e., the rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$.

6 Let r and s be positive integers. Derive a formula for the number of ordered quadruples $(a,\ b,\ c,\ d)$ of positive integers such that

$$3^r \cdot 5^s = \operatorname{lcm}[a, b, c] = \operatorname{lcm}[a, b, d] = \operatorname{lcm}[a, c, d] = \operatorname{lcm}[b, c, d]$$

The answer should be a function of r and s. Note that $\operatorname{lcm}[x, y, z]$ denotes the least common multiple of $x,\ y,\ z$.

7 For which real numbers c is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real x?

8 For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in the simplest form.)

9 Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining fours sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$ with $r,\ s$, and t positive integers.