Some Putnam problems.

Here is a mini-exam for you, based on previous Putnam problems. Give yourself a solid 3 uniterrupted hours - no more - and do what you can.

1. For what region of the real (a, b) plane, do both (possibly complex) roots of the polynomial $z^2 + az + b = 0$ satisfy $|z| < 1$?

2. Define $f_0(x) = e^x$, $f_{n+1}(x) = xf_n'(x)$. Show that $\sum_{n=0}^{\infty} f_n(1)/n! = e^e$.

3. How many real roots does $2^x = 1 + x^2$ have?

4. The real and imaginary parts of z are rational, and z has unit modulus. Show that $|z^{2n} - 1|$ is rational for any integer n.

5. Show that we cannot have 4 binomial coefficients

$$\binom{n}{m}, \binom{n}{m+1}, \binom{n}{m+2}, \binom{n}{m+3}$$

with $n, m > 0$ (and $m + 3 \leq n$) in arithmetic progression.

6. Let $\sum_{n=0}^{\infty} x^n (x-1)^{2n}/n! = \sum_{n=0}^{\infty} a_n x^n$. Show that no three consecutive a_n are zero.

7. Find all possible polynomials $f(x)$ such that $f(0) = 0$ and $f(x^2 + 1) = f(x)^2 + 1$.

8. S is a set with a binary operation $*$ such that (1) $a * a = a$ for all $a \in S$, and (2) $(a * b) * c = (b * c) * a$ for all $a, b, c \in S$. Show that $*$ is associative and commutative.