For the Putnam Group

Big Oh Notation Professor Mel Hausner

This important notation is used to estimate the relative size of functions. $f(x) = O(g(x))$ as $x \to 0$ (read as "$f(x)$ is big Oh of $g(x)$" means that $|f(x)/g(x)|$ is bounded in a neighborhood of 0. More precisely,

$f(x) = O(g(x))$ *as* $x \to 0$ *provided that there exists constants* $M, \delta > 0$ *such that if* $0 < |x| < \delta$*, then* $|f(x)| \leq M|g(x)|$.

If this condition holds for positive $0 < x < \delta$ only, then we say $f(x) = O(g(x))$ as $x \to 0^+$.

In either case, $f(0)$ need not exist. Usually, we take $g(x) > 0$.

A similar notation is used as $x \to \infty$. Here, we say that $f(x) = O(g(x))$ as $x \to \infty$ provided $|f(x)/g(x)|$ is bounded in a neighborhood of ∞. More precisely,

$f(x) = O(g(x))$ *as* $x \to \infty$ *provided that there exists constants* $M, K > 0$ *such that if* $x > K$*, then* $|f(x)| \leq M|g(x)|$.

A typical usage is: $f(x) = x^2 + O(x)$ as $x \to \infty$. This is interpreted as "$f(x)$ equals $x^2 +$ some function which is big Oh of x. We never write $x^2 + O(x) = f(x)$. The big Oh is on the right side of the equation. This is one of the few examples in mathematics where the meaning of the equality sign is fudged.

Some examples:
(1) $f(x) = O(1)$ as $x \to \infty$ means that $f(x)$ is bounded for $x \geq K$ for some K. For example, $\sin x = O(1)$ and $1/x = O(1)$ as $x \to \infty$.

(2) $O(1) + O(1) = O(1)$. In words, the sum of two functions which are bounded is bounded. This is valid for $x \to \infty$ as well as $x \to 0$, and bounded only refers to locally bounded. More generally, $O(g(x)) + O(g(x)) = O(g(x))$.

(3) $(x+1)^2 = x^2 + O(x)$ as $x \to \infty$, since $(x+1)^2 = x^2 + 2x + 1 = x^2 + O(x) + 1 = x^2 + O(x)$, since 1 "gets absorbed" into the $O(x)$. Note however, that $(x+1)^2 = 1 + O(x)$ as $x \to 0$ since $(x+1)^2 = x^2 + 2x + 1 = 1 + O(x) + x^2 = 1 + O(x)$. Here, $x^2 = O(x)$ and gets absorbed into the $O(x)$ term. We could also write $(x+1)^2 = O(x^2)$ as $x \to \infty$, since $x + 1 = O(x)$ and we can "square both sides." So we have the statements
1. $(x+1)^2 = O(x^2)$ as $x \to \infty$
2. $(x+1)^2 = x^2 + O(x)$ as $x \to \infty$
3. $(x+1)^2 = x^2 + 2x + O(1)$ as $x \to \infty$
4. $(x+1)^2 = x^2 + 2x + 1$.

Each is true, but for $n = 1, 2, 3$, statement n is weaker than statement $n + 1$.

A major way of computing estimates of this sort is using the following important result (the Taylor series with remainder):

If $f(x)$ has $n+1$ continuous derivatives in a neighborhood of 0, then for any x in this interval

$$f(x) = f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n! + f^{(n+1)}(c)x^{n+1}/n! \qquad (1)$$

for some c between 0 and x.

Therefore, for such functions f, we have

$$f(x) = f(0) + f'(0)x + f''(0)x^2/2! + /ldots + f^{(n)}(0)x^n/n! + O(x^{n+1}) \text{ as } x \to 0^+ \qquad (2)$$

since $|f^{(n+1)}(c)| \leq \max f^{(n+1)}(y)$ for y between 0 and x. The n-th degree polynomial

$$P_n(x) = f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n! \qquad (3)$$

is called the Taylor polynomial for $f(x)$. In practice, $P_n(x)$ is taken from the power series for $f(x)$. For example, since $1/(1-x) = 1 + x + x^2 + x^3 + \ldots$, we have, for example, $P_2(x) = 1 + x + x^2$, and $1/(1-x) = P_2(x) + O(x^3)$.

To estimate the error in approximating $f(x)$ by $P_n(x)$, we let $M_n = \max|f^{n+1}(y))|$ for $0 \leq |y| \leq |x|$. Then

$$|f(x) - (f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n!)| \leq +M_n x^{n+1}/n!$$

Thus, $|\text{Error}| \leq M_n x^{n+1}/n!$. For example, how good does $1 + x + x^2/2$ approximate e^x near $x = 0$? We have $e^x = 1 + x + x^2/2 + O(x^3)$. We want to investigate $O(x^3)$ a little further. We use equation (1). Here $f(x) = e^x$ so $f''(x) = e^x$. Let's suppose that $0 \leq x \leq 1$. Then $M_2 = \max e^y$ for $0 \leq y \leq x$. Thus, $M_2 = e^x \leq e$. Note also that if we set $m_2 = \min e^y$ for $0 \leq y \leq x$, we have $m_2 = 1$. It follows from (1)that

$$1 + x + x^2/2 + x^3/6 < e^x < 1 + x + x^2/2 + ex^3/6 < 1 + x + x^2/2 + x^3/2$$

where we have used the crude estimate $e < 3$. Averaging the upper and lower bounds, (the average of 1/2 and 1/6 is 1/3) we find that $e^x = 1 + x + x^2/2 + x^3/3 \pm x^3/6$. For example, this gives the estimate $e^{\cdot 1} = 1.105333 \pm .00067$.

A way of converting ∞ to 0 and vice versa is to use the mapping $x \to 1/x$. Thus

$$f(x) = O(g(x)) \text{ as } x \to \infty \text{ if and only if } f(1/x) = O(g(1/x)) \text{ as} x \to 0^+$$

For example, let's estimate $f(x) = \dfrac{x^3}{x-2}$ as $x \to \infty$. Here

$$f(1/x) = \frac{1/x^3}{(1/x) - 2} = \frac{1}{x^2 - 2x^3} = \frac{1}{x^2} \cdot \frac{1}{1-2x} = \frac{1}{x^2}(1 + 2x + 4x^2 + \ldots)$$

Thus, for example $f(1/x) = \dfrac{1}{x^2}(1 + 2x + 4x^2 + O(x^3))$ as $x \to 0$. So, changing x to $1/x$ (and 0 to ∞), we get $f(x) = x^2 + 2x + 4 + O(1/x)$ as $x \to \infty$.