

Putnam Exam: Integration problems

1987B1. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)} dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

1989A2 Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy dx$, where a and b are positive.

1989B3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x) dx$$

(sometimes called the n th moment of f).

a. Express μ_n in terms of μ_0 .

b. Prove that the sequence $\{\mu_n 3^n / n!\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

1990B1. Find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x ((f(t))^2 + ((f'(t))^2) dt + 1990$$

1991 A5. Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2} dx$$

for $0 \leq y \leq 1$.

1992A2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1) \left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992} \right) dy$$

1993A5. Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

1993B4. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 1 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all $x, 0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)dy = g(x) \text{ and } \int_0^1 g(y)K(x, y)dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

1995A2. For what pairs of (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

1997A3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx.$$

2000A4. Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.