Putnam Exam: Sequence problems

1985A3. Let d be a real number. For each integer $m \geq 0$ define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \text{ and } a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

1985B2. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

1987B4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

1990A1. Let

$$T_0 = 2,\ T_1 = 3,\ T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2,\ 3,\ 6,\ 14,\ 40,\ 152,\ 784,\ 5168,\ 40576,\ 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

1992A5. For each positive integer n, let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

for $0 \leq j \leq m-1$.

1992B3. For any pair (x, y) or real numbers, a sequence $(a_n(x,y))_{n\geq 0}$ is defined as follows:

$$\begin{aligned} a_0(x,y) &= x, \\ a_{n+1}(x,y) &= \frac{a_n(x,y)^2 + y^2}{2} \text{ for } n \geq 0. \end{aligned}$$

Find the area of the region $\{(x,y)|(a_n(x,y))_{n\geq 0}$ converges$\}$.

1993A2. Let $(x_n)_{n\geq 0}$ be a sequence of non-zero numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3, \ldots.$$

Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

1997A6. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_n in terms of n and k, $1 \leq k \leq n$.

1999A6. The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

Show that, for all n, a_n is an integer multiple of n.

2001B6. Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

2002A5. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{\frac{a_{n-1}}{a_n} : \ n \geq 1\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots\right\}.$$

2003B2. Let n be a positive integer. Starting with the sequence $1, \ \frac{1}{2}, \ \frac{1}{3}, \ldots, \ \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \ \frac{5}{12}, \ldots, \ \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries and continue until the final sequence produced as a single number x_n. Show that $x_n < \frac{2}{n}$.