Putnam Exam: Number Theory Problems

1988B1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct in $\{2,3,4,\ldots\}$. Show that every composite is expressible as $xy+xz+yz+1$. with $x,\ y$, and z positive integers.

1989A1. How many primes among the positive integers, written in the usual base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1.

1991B4. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

1992A3. For a given positive integer m, find all triples (n,x,y) of positive integers, with n relatively prime to m, which satisfy $(x^2+y^2)^m = (xy)^n$.

1993B1. Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

1994B1. Find all positive integers that are within 250 of exactly 15 perfect squares.

1994B6. For any integer a, set

$$n_a = 101a - 100 \cdot 2^a$$

Show that for $0 \le a,b,c,d \le 99,\ n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a,b\} = \{c,d\}$.

1995A3. The number $d_1d_2\ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\ldots e_9$ is such that each of the nine 9-digit numbers formed by replaying just one of the digits d_i in $d_1d_2\ldots d_9$ by the corresponding digits e_i $(1 \le i \le 9)$ is divisible by 7. The number $f_1f_2\ldots f_9$ is related to $e_1e_2\ldots e_9$ in the same way: that is, each of the nine numbers formed by replaying one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i,, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2\ldots d_9 = 199501996$ then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

1996A5. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

1997A5. Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

1997B5. Prove that for $n \geq 2$,

$$\left.2^{2^{\cdot^{\cdot^{2}}}}\right\}n \equiv \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}n-1 \pmod{n}.$$

1998A4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998B4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

1999B6. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

2000A2. Prove that there exist infinitely many integers n such that $n,\ n+1,\ n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2,\ 1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

2000B2. Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

2001A5. Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

2002B5. A palindrome in base b is a positive integer whose base-b digits read the same backwards and forward; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

2002B6. Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say that two integer polynomials of congruent mod p if corresponding coefficients are congruent modulo p.)

2004B3. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.