Putnam Exam: Combinatorics Problems

1985A1. Determine, with proof, the number of ordered triples $\{A_1, A_2, A_3\}$ of sets which have the property that
(i) $A_i \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$,
where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where $a,\ b,\ c$, and d are nonnegative integers.

1985B3. Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be a doubly infinite array of positive integers, and suppose that each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

1989B4. Can a countably infinite set have an uncountable collection of nonempty subset such that the intersection of any two of them is finite?

1990A6. If X is a finite set, let $|X|$ denote the number of elements of X. Call an ordered pair (S, T) of subsets of $\{1, 2, \dots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \dots, 10\}$ are there? Prove your answer.

1991A6. Let $A(n)$ denote the number of sums of positive integers $a_1 + a_2 + \cdots + a_r$ which add up to n with $a_1 > a_2 + a_3,\ a_2 > a_3 + a_4, \dots, a_{r-2} > a_{r-1} + a_r,\ a_{r-1} > a_r$. Let $B(n)$ denote the number of $b_1 + b_2 + /cdots + b_s$ which add up to n, with
(i) $b_1 \geq b_2 \geq \cdots \geq b_s$.
(ii) each b_i is in the sequence $1, 2, 4, \dots, g_j, \dots$ defined by $g_1 = 1.\ g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
(iii) if $b_1 = g_k$ then every element if $\{1, 2, 4, \dots, g_k\}$ appears at least once as a b_i.
Prove that $A(n) = B(n)$ for each $n \geq 1$.

(For example, $A(7) = 5$ because the relevant sums are $7,\ 6+1,\ 5+2,\ 4+3,\ +2+1$, and $B(7) = 5$ because the relevant sums are $4+2+1,\ 2+2+2+1,\ 2+2+1+1+1,\ 2+1+1+1+1+1,\ 1+1+1+1+1+1+1$.)

1992B1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

1993A3. Let P_n be the set of subsets $\{1, 2, \dots, n\}$. Let $c(n, m)$ be the number of functions

$f{:}P_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

1993A4. Let $x_1,\ x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1,\ y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

1993B6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$ and replace them with $2x$ and $y - x$.

1994A6. Let $f_1,\ f_2, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$F = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdot \circ f_{10}^{e_{10}}\}$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A is any nonempty finite set of integers, then at most 512 of the functions in F map A to itself.

1995A1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

1995B1. For a partition π of $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

1996A3. Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

1996A4. Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that
(1) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
(2) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ [for $a,\ b\ c$ distinct];
(3) (a, b, c) and (c, d, a) are both is S if and only if (b, c, d) and (d, a, b) are both is S.
Prove that there exists a one-to-one function g from A to $\mathbf{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

1996B1. Define a *selfish set* to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1,\ 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

1996 B5. Given a string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example $\Delta(XOOXOOX) = -1$. We call a string S *balanced* if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

2000B1. Let $a_j,\ b_j,\ c_j$ be integers for $1 \leq j \leq N$. Assume for each j, at least one of $a_j,\ b_j,\ c_j$ is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ value of $j,\ 2 \leq j \leq N$.

2000B5. Let S_0 be a finite set of positive integers. We define finite sets $S_1,\ S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

2001B1. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

2003A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: $4,\ 2+2,\ 1+1+2,\ 1+1+1+1$.