For the Putnam Group

Important Series You Gotta Know!

Professor Mel Hausner

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$	$= \sum_{n=0}^{\infty} x^n$	Geometric Series ($\lvert x\rvert < 1$)
$\frac{1}{1+x} = 1 - x + x^2 - \cdots$	$= \sum_{n=0}^{\infty} (-1)^n x^n$	Geom. Ser. (variant), ($\lvert x\rvert < 1$)
$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$	$= \sum_{n=0}^{\infty} (n+1)x^n$	Derivative of Geom. Ser. $\lvert x\rvert < 1$)
$\frac{1}{(1-x)^{k+1}} = 1 + (k+1)x + \cdots$	$= \sum_{n=0}^{\infty} \binom{n+k}{k} x^n$	k-th deriv. of Geom. Ser. $\lvert x\rvert < 1$)
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	Logarithm Series ($-1 < x \leq 1$)
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	Arctan Series ($-1 < x \leq 1$)
$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$	$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$	Exponential Series (all x)
$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$	$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	Sine Series (all x)
$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$	$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	Cosine Series (all x)

In addition, we have the binomial theorem, valid for all real values of α and $|x| < 1$.

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

The Binomial Theorem is a finite sum when α is a non-negative integer, and in that case it naturally converges for all x. Here, as in the 4th sum above,

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

A Few Infinite Sums:

1. $\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} = \ln 2$
2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

A Few Definite Integrals:

1. $\int_0^\infty x^n e^{-x} dx = n!$
2. $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$
3. $\int_0^{\pi/2} \sin^{2n} x dx = \frac{1}{2^{2n}} \binom{2n}{n} \frac{\pi}{2}$