Basic Facts about Binomial Coefficients

There are many equivalent ways of defining $\binom{n}{r}$. (Read this as "n choose r.") Here we assume $0 \le r \le n$.[1] Here are four of doing this.

1. **The Factorial Formula:**

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This is enough to give the basic identities

$$\binom{n}{0} = \binom{n}{n} = 1; \ \binom{n}{r} = \binom{n}{n-r}$$

2. **Recursion on r (Pascal's triangle):**

$$\binom{0}{0} = 1; \ \binom{0}{r} = 1 \text{ for } r \neq 0$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \ (r > 0)$$

These formulas lead directly to the familiar Pascal Triangle, where each entry below the top is the sum of the one above and the one above and one over to the left.

	$r =$ 0	1	2	3	4	5	6
$n = 0$	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Pascal's Triangle

Suggestion. Memorize each row of this table! All educated people should know it.

[1] If $r < 0$ or $r > n$, we have $\binom{n}{r} = 0$.

3. **Combinatoric Method:**
By definition, $\binom{n}{r}$ is the number of r-subsets of an n-set. By definition, an n-set is a set consisting of n elements, and similarly for an r-subset. Thus, for example $\binom{4}{2}$ is the number of 2-subsets of a 4-set. If the 4-set consists of 1,2,3,4, the 2 subsets are 12, 13, 14, 23, 24, 34. Since there are 6 in all, we have $\binom{4}{2} = 6$ This definition shows why we read $\binom{n}{r}$ as n *choose.* r. We are choosing r things from among n.

4. **Binomial Theorem Result:**

$$(1+x)^n = 1 + nx + \cdots + \binom{n}{r}x^r + \cdots + nx^{n-1} + x^n = \sum_{r=0}^{n} \binom{n}{r} x^r \tag{1}$$

For example (see row 5 in the Pascal Triangle)

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Because of the binomial theorem, the numbers $\binom{n}{r}$ are also called *binomial coefficients*. Other notations, used less frequently are $C(n,r)$, ${}_nC_r$, and C_r^n.

All of these 4 definitions are equivalent. That is, if we used any one of these results as the definition of $\binom{n}{r}$, the other results would follow. Some results to know:

$$\binom{n}{0} = \binom{n}{n} = 1 \tag{2}$$

$$\binom{n}{r} = \binom{n}{n-r} \tag{3}$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ (Pascal's Identity)} \tag{4}$$

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} \tag{5}$$

For example, to compute $\binom{50}{47}$ we use (3) and (5) to get

$$\binom{10}{6} = \binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} = 210$$

Many identities involving binomial coefficients (called binomial identities) can be found using the binomial theorem (1). For example, putting $x = 1$ in (1), we get

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Similarly, using $x = -1$, we get

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0 \ (n > 0).$$

(Explain why we need $n > 0$ here.)

Similarly, differentiating (1) and then putting $x = 1$ gives

$$\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1}.$$

Here are a few problems to work on.

1. Write $\sum_{r=0}^{n} r^2 \binom{n}{r}$ in closed form.

2. As above for $\sum_{r=0}^{n} \frac{1}{r+1} \binom{n}{r}$.

3. Compute $\sum_{r=0}^{n} \binom{n}{2r}$.

4. Compute $\sum_{r=0}^{n} \binom{n}{r}^2$ in closed form.

5. Write $\binom{n}{r}$ as a sum of terms involving $\binom{n-3}{s}$ $(n \geq 3)$. Hint: Use (4).

6. Generalize Exercise 5, using terms involving $\binom{n-k}{s}$ $(n \geq k)$ for fixed k.