Putnam 1991, B5
Here is an alternate approach to the problem:
Given

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y) \\ g(x+y) &= g(x)f(y) + f(x)g(y) \end{aligned}$$

with f, g non-constant, real valued and differentiable and $f'(0) = 0$. To prove

$$(f(x))^2 + (g(x))^2 = 1$$

(Anticipating f and g as cosine and sine,) set

$$z(x) = f(x) + ig(x)$$

It is then easy to verify that the given system is equivalent to

$$z(x+y) = z(x)z(y) \tag{1}$$

Now take the derivative with respect to y and set $y = 0$:

$$z'(x) = cz(x) \text{ where } c = z'(0)$$

But this differential equation has the solution $z = Ae^{cx}$. (For a proof, set $w = ze^{-cx}$ and check that $w' = 0$.) But using (1) this gives $A^2 = A$. So $A = 0$ or $A = 1$. Reject $A = 0$ beause this give $f = g = 0$ and we are given that these are not constant. So $A = 1$ and so $z(x) = e^{cx}$. But $c = z'(0) = f'(0) + ig'(0) = ik$, where k is real, since $f'(0) = 0$. Thus,

$$z(x) = e^{ikx} = \cos kx + i \sin kx$$

and so $f(x) = \cos kx$ and $g(x) = \sin kx$. This clearly implies the conclusion (and in fact gives the precise form of f and g.)