Solution problem B4, 2003. We have $P(z) = (z - r_1)(z - r_2)(z - r_3)(z - r_3)$ has rationalrcoefficients. We are given that $r_1 + r_2$ is rational, and that $r_1 + r_2 \neq r_3 + r_4$. We must show that $r_1 r_2$ is rational.

In what follows, we denote rational numbers by $q_1, q_2, \ldots$ Relating the coefficients to the roots r_i, we have

$$\sigma_i = q_i; (1 = 1, 2, 3, 4)$$

Here σ_i is the i-th elementary function of the roots:

$$\sigma_1 = r_1 + r_2 + \ldots;\ \sigma_2 = r_1 r_2 + r_1 r_3 + \ldots; \sigma_3 = r_1 r_2 r + 3 + \ldots;\ \sigma_4 = r_1 r_2 r_3 r_4$$

Now to the problem. Let

$$S = r_1 + r_2;\ T = r_3 + r_4;\ U = r_1 r_2;\ V = r_3 r_4$$

We are given $S = q_5$ (that is, rational) and $U \neq V$. We have to show that U is rational. We cazn express all of the σ_i in terma of S, T, U, V. Thus
(1) $S + T = \sigma_1 = q_1$
(2) $ST + U + V = \sigma_2 = q_2$
(3) $TU + SV = \sigma_3 = q_3$
(4) $UV = \sigma_4 = q_4$

Since $S = q_5$, (1) gives $q_5 + T = q_1$, and solving for T, we get $T = q_6$. Substituting in (2) this gives $q_5 q_6 + U + V = q_2$, yielding
(5) $U + V = q_7$
Substituting in (3) gives
(6) $q_6 U + q_5 V = q_3$

Now *solve (5) and (6)* for U and V. Here we must use $q_5 \neq q_6$ (ie $S \neq T$). Eliminate V by muliplying (5) by q_5 and subtracting. This gives
(5') $q_5 U + q_5 V = q_5 q_7$, and subtracting (5') from (6), we get
(7) $(q_6 - q_5)U = q_8$, so $U = q_9$, which is the result.

A general, after the fact observation: Why was this so hard (for me) in lecture, and why so easy in the comfort of my home? Answer: It's not that I'm nervous in front of a class, or can't think on the spot. It's because in lecture, I used $S \in Q$, while in this note, I used $s = q_5$. Isn't it much easier to use equality than $\in$? I think so. We are reduced to solving two linear equations ((5) and (6)) in 2 unknowns. No problem!