Introduction to Mathematical Modeling, Spring 2000

Assignment 1, due February 1.

1. In quantum mechanics there arises the problem of finding all numbers, p, that satisfy the equation

$$p = \sin(Lp) \ .$$

Of course, $p = 0$ is a solution. The other solutions are called "nontrivial".

a. Give a graphical argument that there are nontrivial solutions if and only if $|L| > 1$.

b. Give a graphical argument that when L is very large, the smallest positive solution is roughly $p \approx \pi/L$. This may be easier after you do part c.

c. To find this smallest positive solution more accurately, *rescale* the equation by setting $u = Lp$ and $\lambda = 1/L$. Write the equation in terms of the unknown u and parameter λ.

d. Write the first two *non zero* terms of the Taylor series expansion of $\sin(u)$ about $u_0 = \pi$.

e. If $u(\lambda) \approx u_0 + \lambda u_1 + \lambda^2 u_2$, determine u_0, u_1, and u_2 (hint: $u_0 = \pi$) by substituting into the equation from part c. and matching powers λ^0, λ^1, and λ^2. This will take much less time if you are careful not to compute things you don't need.

2. Recall the equation $x + \epsilon x^2 = 2$ with $\epsilon = .1$

a. Review the computation from class which showed that $x \approx 2 - 4\epsilon + 16\epsilon^2$. Write out the algebra to hand in.

b. Review the less systematic approach that lead to $x_0 = 2$, $x_1 = 2 - \epsilon x_0^2 = 1.6$, $x_2 = 2 - \epsilon x_1^2$, and so on.

c. in Matlab (or a reasonable alternative, but not just using a calculator), compute the approximations from part a. and see how accurate they are with $\epsilon = .1$. This involves computing the exact solution and subtracting it from the approximate solution.

d. Compute the first ten approximations from part b., $x_k + 1 = 2 - \epsilon x_k^2$. Plot the differences $x_k - x$ on a log scale. What do you notice about the plot?

3. Download the matlab program on this web page and look at the plots it has made. Bifurcations are places where, moving from left to right, one curve splits into two curves. The bifurcation points are the values of a where there is a bifurcation. What patterns can you notice about the successive bifurcations?