Assignment 9, due November 18

Corrections: [none yet]

1. Some stochastic interest rate models involve only the *short rate* process. If R_t is the short rate at time t, then if you borrow w at time t, at time $t+dt$ you pay $w(1+R_t)$. If you pay only the interest, then at time $t+dt$ you still owe w. It often is convenient to set $w=1$. If necessary you can multiply by w at the end. A simple short rate "equilibrium model" is $dR_t = -\gamma(R_t - r_0)dt + \sigma dW_t$. This says that R_t on average "reverts" to its mean value r_0 with exponential rate γ but is driven away from the equilibrium with a noise with strength σ. Suppose $\rho > 0$ is a discount rate. Let $g(r,\rho)$ be the expected total discounted payment, assuming that $R_0 = r$.

   (a) Write a definition of $g(r)$ as the expectation of a discounted additive functional.

   (b) Identify the generator of the process and use this to write the differential equation g satisfies.

   (c) Write the solution of the equation in part (b). *Hint*: Guess. Try exponentials, polynomials, exponentials of polynomials, polynomials times exponentials, etc.

   (d) Which parameter in the problem does g not depend on? Why not?

   (e) Is it true that $g(r,\rho) \to \infty$ as $\rho \to 0$? Explain this result in terms of the behavior of the process.

2. Consider a Brownian motion X_t starting at $X_0 = x$ with $|x| < M$. Define T to be the first time $|X_t| = M$.

   (a) Write the differential equation that is satisfied by

   $$g(x) = \mathrm{E}_x\left[\, e^{-rT} \right] \ .$$

   Hint: You can formulate the functional in a way closer to what we did in class using $f(x,t) = \mathrm{E}_{x,t}\left[\, e^{-r(T-t)} \right]$. This does not depend on t, but the t might help with the derivation. What boundary condition does g satisfy at $x = \pm M$?

   (b) Write the solution, as a function of x and r. *Hint:* Guess. Try using a cosine or some exponentials to help build the solution.

   (c) Does $g(x,r)$ "blow up" as $r \to 0$? What does this tell you about hitting times? *Hint:* The answer to part (d) is helpful.

(d) Suppose T is a random variable with $\mathrm{E}[e^{\rho T}] = A < \infty$ with $\rho > 0$. Define the *tail probabilities* $Q(t) = \Pr(T > t)$. Show that the tail probabilities are exponentially small. *Hint:* Let $\mathbf{1}_t(T) = 1$ if $T > t$ and zero otherwise. Why is it that (this is a "standard math trick")

$$Q(t) = \mathrm{E}[\mathbf{1}_t(T)] = \mathrm{E}[\mathbf{1}_t(T) e^{\rho T} e^{-\rho T}] \leq \mathrm{E}[\mathbf{1}_t(T) e^{\rho T} e^{-\rho T}] \ ?$$

3. Suppose T is a random variable and $S(\rho) = \mathrm{E}\left[\, e^{\rho T} \right]$ is the *moment generating function.* The n^{th} moment of T is $m_n = \mathrm{E}[\, T^n]$.

   (a) Show that $m_n = \partial_\rho^n S(0)$. This is how the moment generating function generates moments.

   (b) Use the answer to Exercise 2 to evaluate $\mathrm{E}_x[T]$ and $\mathrm{var}_x(T)$.

   (c) Show that $\mathrm{E}_x[T]$ is a value function that satisfies a natural differential equation involving the generator of Brownian motion and suitable boundary conditions. Do not try to get the variance formula in this direct way. It is more complicated.