Assignment 6, due October 28

Corrections: [none yet]

1. For each part of this exercise, find an ortho-normal basis for a subspace of $\mathcal{S} \subset \mathbb{R}^4$. Each of these is defined as an orthogonal complement of some space. State the number of basis vectors you need and find them. The answer is not unique. The purpose is to practice thinking of orthogonal complements as vector spaces, as in the variational formulation of the symmetric eigenvalue/eigenvector problem.

   (a) $\mathcal{S} \subset \mathbb{R}^4$ is the orthogonal complement of $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

   (b) $\mathcal{S} \subset \mathbb{R}^4$ is the orthogonal complement of $u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

2. Suppose A and B are matrices, not necessarily square. Show that if AB is square then BA is square and $\text{Tr}(AB) = \text{Tr}(BA)$.

3. Suppose C is a symmetric matrix with eigenvalues λ_j. Show that

$$\sum_{jk} c_{jk}^2 = \sum_j \lambda_j^2 \ .$$

4. Suppose f_t and g_t are continuous functions of t, not random. Define

$$X_f^{\Delta t} = \sum_{t_k < t} f_{t_k} \left(W_{t_{k+1}} - W_{t_k} \right) \ .$$

   and similarly for $X_g^{\Delta t}$. Use the usual definition $t_k = k\Delta t$. Let $p(x_f, x_g, \Delta t)$ be the joint PDF of $(X_f^{\Delta t}, X_g^{\Delta t})$. Show that as $\Delta t \to 0$, the joint density converges to $p(x_f, x_g)$, which is Gaussian with mean zero. Evaluate the covariance matrix $\text{cov}(X_f, X_g)$. The entries are integrals involving f_t and g_t. This is the definition of

$$X_{f,t} = \int_0^t f_t dW_t \ .$$

5. Consider the linear SDE $dX_t = AX_t dt + BdW_t$, where W_t is a multi-variate standard Brownian motion. Define the *fundamental solution*, $S(t)$, by

$$\frac{d}{dt}S(t) = AS(t) = S(t)A \ , \quad S(0) = I \ .$$

Define

$$X_t = S(t)X_0 + \int_0^t S(t-t')B \, dW_{t'} \ .$$

[The fundamental solution happens to commute with A, which you can verify.]

   (a) Show that this X satisfies the linear SDE. *Hint:* There is a lot to do here. Part of the problem is to decide what you need to do. Assume that different components of W_t are independent, so integrals involving them are also independent. Try to use matrix/vector notation as much as possible – it will be simpler that way.

   (b) Write an integral involving $S(t)$ that is equal to $\text{cov}(X_t$.

   (c) Assume that

$$\int_0^\infty \|S(t)\| \, dt < \infty \ .$$

   Show that $C = \lim_{t\to\infty} \text{cov}(X_t)$ exists and write an integral formula for it.

   (d) Show (using the integral formula from part (c) that C satisfies the Lyapunov equation $AC + CA^t + BB^t = 0$ and that C is positive definite.

   (e) The Ornstein Uhlenbeck process is the one component process $dX_t = -\gamma X_t dt + \sigma dW_t$. We calculated the transition density $G(y, x, t)$ for this in an earlier assignment. Here $G(y, x, t)$ is the PDF of X_t, as a function of x, with the condition that $X_0 = y$. Show that if $\gamma > 0$, then the PDF of X_t converges to a mean zero Gaussian with the variance predicted by part (d).

6. Suppose $X \sim \mathcal{N}(0, C)$ in d dimensions. Consider unit vectors $u_k \in \mathbb{R}^d$, $\|u_k\|^2 = u_k^t u_k$, for $k = 1, \ldots, d$. Define the *scalars* (one component random variables) $Y_k = u_k^t X$ for $k = 1, \ldots, d$.

   (a) Show that $\text{cov}(Y_1, \ldots, Y_d) = \Lambda$ if and only if the u_k are the eigenvectors of C with $Cu_k = \lambda_k u_k$. Here $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d)$. [This is the diagonal matrix with λ_k in spot k on the diagonal.] *Hint:* Show you can work in a coordinate system in which C is diagonal.

   (b) Show that if scalars $Z_k = v_k^t X$ are independent, and if $k = 1, \ldots d$, then the v_k are eigenvectors.

   (c) Show that this is not true if there are fewer than d vectors v_k. It suffices to give just one counter-example. That can be two vectors v_1 and v_2 in $\mathbb{R}^3$ with some 3×3 covariance matrix C.

7. Consider the "vector" $W_{[0,T]}$. This is the Brownian motion path over the time interval $[0,T]$. Consider vectors (functions of t) of the form

$$v_{k,t} = \sin(\omega_k t) \; .$$

Define the scalar random variables

$$Y_k = \int_0^T v_{k,t} dW_t \; .$$

Show that if ω_k is the sequence $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\ldots$, then the Y_k are independent. This, by itself, does not show that the functions v_k are principal components, but they are.

Computing exercise

None this week.