**Stochastic Calculus**, Courant Institute, Fall 2019
http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2019/StochCalc.html
**Always** check the `classes` message board before doing any work on the assignment.

# Assignment 11, due December 2

**Corrections:** [none yet]

1. Let $N_t$ be the counting function for a Poisson arrival process with rate $\lambda$. Show that $X_t = N_t - \lambda t$ is a martingale.

2. Let $\mathcal{F}_t$ be the filtration (family of $\sigma-$algebras) generated by $W_{[0,t]}$. Suppose $T > t$.
   (a) Calculate $\mathrm{E}\left[\, W_T^2 \mid \mathcal{F}_t \right]$.
   (b) Let $Q$ be the random variable
   $$Q = \int_0^T W_s dW_s \ .$$
   Calculate $\mathrm{E}[\, Q \mid \mathcal{F}_t]$.

3. Define the random variable
   $$Q = \int_0^T W_s ds \ .$$
   Let $\mathcal{G}_t$ be the $\sigma-$ algebra generated by the value $W_t$. Note that $\mathcal{G}_t$ does not "know" the value of $W_s$ for any $s \neq t$, except that $W_0 = 0$. The family $\mathcal{G}_t$ is not a filtration, because $\mathcal{G}_{t_2}$ does not contain $\mathcal{G}_{t_1}$ if $t_2 \neq t_1$. Describe the random variable
   $$R_t = \mathrm{E}[\, Q \mid \mathcal{G}_t] \ .$$
   Describe $R_t$ in terms of $W_t$ and a random variable independent of $R_t$. *Hint:* $Q$ and $W_t$ are jointly normal with variances and covariance that you can calculate.

4. Consider the random variable $S_t = s_0 e^{\sigma W_t + at}$. Find the value of $a$ so that $S_t$ satisfies the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$. Do this using Ito's lemma applied to some function $f(W_t, t)$. Write a formula for $f(w,t)$ and calculate the partial derivatives involved.

5. Suppose $A_n$ is a family of random variables with
   $$\sum_{n=1}^{\infty} \mathrm{E}\left[\, A_n^4 \right] < \infty \ .$$
   Use the Borel Cantelli lemma from the Class 11 notes to show that
   $$A_n^4 \to 0 \ \text{ as } \ n \to \infty \ , \ \text{ almost surely } \ .$$
   Conclude that $A_n \to 0$ as $n \to \infty$ almost surely.

6. Let $X_k$ be independent and identically distributed random variables with

$$\mathrm{E}[\,X_k] = 0\;,\;\; \mathrm{E}\big[\,X_k^2\big] = \sigma^2 < \infty\;,\;\; \mathrm{E}\big[\,X_k^4\big] = \mu_4 < \infty\;.$$

Let the sample mean up to $n$ is

$$A_n = \frac{1}{n}\sum_{k=1}^{n} X_k\;.$$

Find a formula for $\mathrm{E}\big[\,A_n^4\big]$ in terms of $\sigma_2$ and $\mu_4$. Use this formula to show that $A_n \to 0$ as $n \to \infty$ almost surely using the method of Exercise 6. [This is a proof of the almost sure law of large numbers assuming a finite fourth moment. Kolmogorov gave a proof that allows $\mu_4 = \infty$ but requires finite variance. That proof is harder but not very hard. Then he gave a spectacular proof assuming only that $\mathrm{E}\,[|X_k|] < \infty$. This is the *Kolmogorov strong law* of large numbers.]

7. Define the quadratic variation of Brownian motion as

$$[W]_t = \lim_{m\to\infty} \sum_{t_k<t} \left(W_{t_{k+1}} - W_{t_k}\right)^2\;.$$

Use the definitions from the Class 11 notes: $\Delta t = 2^{-m}$ and $t_k = k\Delta t$. Prove that the limit exists almost surely and $[W]_t = t$. *Hint:* the mean on the right converges to $t$ and the variance goes to zero fast enough to apply the Borel Cantelli lemma.