Assignment 10, due November 25

Corrections: [none yet]

1. Consider a Poisson arrival process on an interval $(0, t)$ with intensity λ. Suppose we divide the interval into M pieces of equal length Δt, and put a "hit" in each interval with probability $\lambda \Delta t$ (at most one hit per interval, all intervals independent). Let N_t be the (random) number of hits in total.
   (a) Write an exact formula for $P_n(t, \lambda, M) = \Pr(N_t = n)$ using the binomial formula from basic probability.
   (b) Keep n and t fixed and evaluate
   $$P(t, \lambda) = \lim_{M \to \infty} P(t, \lambda, M) \; .$$
   This should agree with the formula for the same quantity in the notes.

2. Let $T_k^{(1)}$ and $T_k^{(2)}$ be the Poisson arrival times for independent processes with rates λ_1 and λ_2 respectively. Let T_k be the union of these two processes. That is, T_k is the sequence that consists of all the arrivals $T_k^{(1)}$ and $T_k^{(2)}$. Show that T_k is a Poisson arrival process with rate $\lambda_1 + \lambda_2$.
   (a) Do this in a simple way by considering the probability that there is a $T_k^{(1)}$ or $T_k^{(2)}$ arrival in an interval of length Δt,.
   (b) Do this in a complicated way by letting $S^{(1)}$ and $S^{(2)}$ be independent inter-arrival times for $T_k^{(1)}$ and $T_k^{(2)}$. Let $S = \min(S^{(1)}, S^{(2)})$ and show S has the inter-arrival distribution for rate $\lambda_1 + \lambda_2$.

3. Suppose there is a payout that depends on N_T, where N_t is the counting process for a Poisson process with intensity λ. Define the value function $f(n, t) = \mathrm{E}_{n,t}[V(N_T)]$. Let g be a function of a non-negative integer n. This could be written as a sequence $g_0, g_1, \ldots$, but it is convenient here to use function notation $g(n)$.
   (a) How does the generator of the Poisson process act on g? More precisely, if $h = Lg$, write a formula for $h(n)$ in terms of g.
   (b) Use this to write the backward in the form
   $$\frac{df(n,t)}{dt} = \ldots \; .$$
   (c) Find a formula for $f(n, t)$ if $V(n) = n$. *Hint:* N_t is the number of arrivals up to time t. What is the probability of a new arrival in time dt? Show that your formula satisfies the backward equation.

(d) Find a formula for $f(n,t)$ if $V(n) = 0$ for $n \geq k$ and $V(n) = 1$ for $n < k$. This can be constructed from the known formula for $\Pr(T_j \geq s)$ and the "initial condition" that $N_t = n$. Verify that this formula satisfies the backward equation.

(e) Suppose the inner product is

$$\langle q, g \rangle = \sum_{k=0}^{\infty} q_k g_k \ .$$

Use this to find a formula for $L^* q$ and to find the differential equation

$$\frac{dp_n(t)}{dt} = \cdots \ .$$

Here $p_n(t) = \Pr(N_t = n)$.

(f) Derive the differential equations for $p_n(t)$ of part (e) directly using $\Pr(n-1 \to n \text{ in time } dt)$. More formally,

$$\Pr(n-1 \to n \text{ in time } dt) = \Pr(N_{t+dt} = n \mid N_t = n-1) \ .$$

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4. Suppose $\mathcal{R}$ be the collection of all rectangles in $\mathbb{R}^2$. A rectangle is a set $B = \{(x,y) | a \leq x \leq b \text{ and } c \leq y \leq d\}$. Let $\mathcal{F}$ be the $\sigma-$algebra generated by $\mathcal{R}$. Show that the following sets are in $\mathcal{F}$.

   (a) $D = \left\{(x,y)|x^2 + y^2 < R^2\right\}$. This is the disk of radius R centered at the origin. *Hint:* It is possible to arrange the rational points ("rational point" = point with rational coordinates) in D in a sequence (countable list). Every point in D is in some rectangle about a rational point. Explaining all of this can take a while.

   (b) $A = \left\{(x,y)|r^2 \leq x^2 + y^2 < R^2\right\}$. This is an annulus. *Hint:* use part (a), complements, and intersections. Explaining this is easy.

   (c) $C = \left\{(x,y)|x^2 + y^2 = R^2\right\}$. This is the unit circle. *Hint:* this is an intersection of a countable sequence of annuli (one "annulus", two "annuli").

5. In $\mathbb{R}^2$, let $\mathcal{F}$ be the $\sigma-$algebra generated by the function $r^2(x,y) = x^2+y^2$. Suppose $(X,Y) \in \mathbb{R}^2$ is Gaussian with distribution $\mathcal{N}(0,I)$. Let $f(x,y) = x^2$. Find an expression for $g(X,Y) = \mathrm{E}[\, f(X,Y)|\mathcal{F}]$.