Jonathan Goodman, October, 2018

Final exam practice

Information

- The final exam is Monday, December 17 in room 109 from 7:10 to 9pm.
- The exam starts promptly at 7:10, don't be late.
- You are allowed one standard size ($8\frac{1}{2}'' \times 11''$) sheet of paper with any information you like. No other information or electronics are allowed.
- Write all answers in one or more blue books provided. Hand in only the blue books.
- Write your name on each blue book and number them (e.g. 1 of 1, 2 of 3 etc.)
- You will receive 20% credit for question if you write nothing.
- Anything you do write may be counted against you if it is wrong.
- Cross out anything you think is wrong. If you have two answers, the wrong one will count against the right one.
- On multiple choice or true/false questions, give a few words or sentences of explanation. You may lose points even with a correct answer, if it isn't explained.
- Suppose that W_t is standard Brownian motion and $dX_t = W_t^2 dt$. Evaluate the quadratic variation

Practice questions

True/False

1. If X_t is a stochastic process with $\mathrm{E}[\Delta X | \mathcal{F}_t] = O(\Delta t)$, then X_t is a diffusion process.
2. If X_t is a Markov process and $Y_t = f(X_t)$ for some function $y = f(x)$, then Y_t is a Markov process.
3. If X_t is stochastic process with $\mathrm{E}[\Delta X] = O(\Delta t)$ and $\mathrm{E}[\Delta X^2] = O(\Delta t)$, then $\mathrm{var}(\Delta X) = \mathrm{E}[\Delta X^2] + O(\Delta t^2)$.
4. If X is a random variable with $\mathrm{E}[X^2] < \infty$, then $\mathrm{E}\,|X|] < \infty$.

5. If W_t is Brownian motion, then $X_t = \int_0^t f(s) dW_s$ is Gaussian. Here $f(s)$ is a fixed deterministic function.

6. If W_t is Brownian motion, then $X_t = \int_0^t a(W_s) dW_s$ is Gaussian. Here $a(w)$ is a fixed deterministic function.

7. If X_t is a Markov process then $Y_t = \int_0^t a(s) dX_s$ is a Markov process.

8. If X_t is a diffusion process and $\frac{d}{dt} E[X_t] = 0$, then X_t is a martingale.

9. If $S_{1,t}$ and $S_{2,t}$ are geometric Brownian motions, then $S_t = S_{1,t} + S_{2,t}$ is a geometric Brownian motion. Hint: this does not depend on whether S_1 and S_2 are correlated, as long as the correlation coefficient has $|\rho| < 1$.

Multiple choice

1. Suppose $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions. Which of the following processes is not a Markov process

   (a) $X_t = W_{1,t} + W_{2,t}$
   (b) $X_t = W_{1,t}^3 - 3tW_{1,t}$
   (c) $X_t = W_{1,t} W_{2,t}$
   (d) $X_t = W_{1,t}^2 + W_{2,t}^2$.

2. Suppose $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions. Which of the following processes is not a martingale

   (a) $X_t = W_{1,t} + W_{2,t}$
   (b) $X_t = W_{1,t}^3 - 3tW_{1,t}$
   (c) $X_t = W_{1,t} W_{2,t}$
   (d) $X_t = W_{1,t}^2 + W_{2,t}^2$.

3. Suppose X_t is a diffusion process with infinitesimal mean a_t and infinitesimal variance μ_t. Which hypotheses imply the formula

$$\int_0^t X_s dX_s = \frac{1}{2} X_t^2 - \frac{1}{2} t \ ?$$

   (a) $\mu_t = 1$ and $a_t = 0$
   (b) $a_t = 0$
   (c) $\mu_t = 1$
   (d) The formula is true for any diffusion.

Full answer questions

1. If X_t is defined by

$$X_t = \int_0^t s^2 W_s dW_s$$

   calculate $\mathrm{var}(X_t)$.

2. Suppose $X_t = W_t^3$ and W_t is standard Brownian motion. Write the SDE that X_t satisfies.

3. Suppose that $X_t = W_{t^2}$ and W_t is standard Brownian motion. Show that X_t is a diffusion and find its infinitesimal mean and variance.

4. Suppose that
$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty \ .$$
Suppose that X_k is a family of random variables, not necessarily independent, with mean zero and variance σ_k^2. Show that the following infinite sum exists and is finite almost surely.
$$\sum_{k=1}^{\infty} X_k^2 \ .$$

basket

5. Suppose $S_{1,t}$ and $S_{2,t}$ are geometric Brownian motions that satisfy $dS_{1,t} = \sigma_1 S_{1,t} dW_{1,t}$ and $dS_{2,t} = \sigma_2 S_{2,t} dW_{2,t}$. Suppose $W_{1,t}$ and $W_{2,t}$ are possibly correlated Brownian motions with correlation coefficient ρ. Show that $X_t = S_{1,t} S_{2,t}$ is a martingale if and only if $\rho = 0$.

6. In the setting of problem basket 5, find an expression for $\mathrm{E}[S_{1,T} S_{2,T}]$. Do this by writing a formula for $S_{1,T}$ in terms of $W_{1,T}$ and $S_{2,T}$ in terms of $W_{2,T}$. Then express $W_{1,T}$ and $W_{2,T}$ as linear combinations of two independent mean zero Gaussians (there are many ways to do this). Then use the fact that $\mathrm{E}[e^{\mathcal{N}(a,v)}] = e^a e^{\frac{1}{2}v^2}$

7. Suppose that W_t is standard Brownian motion and $dX_t = W_t^2 dt$. Evaluate the quadratic variation
$$Q_t = \lim_{n\to\infty} \sum_{t_k < t} (X_{t_{k+1}} - X_{t_k})^2 \ .$$
Here $\Delta t = 2^{-n}$ and $t_k = k\Delta t$.

8. Suppose $dS_t = \mu S_t dt + \sigma S_t dW_t$. Evaluate $f(s,t) = \mathrm{E}[S_t^p]$. Assume $t \le T$ and $p > 0$. Write the backward equation and final condition. Assume the solution has the form $f(x,t) = A(t)s^p$, find the differential equation A satisfies, and then find a formula for $A(t)$ using the final condition.

9. Suppose $dX_t = adt + \sigma dW_t$ (Brownian motion with constant drift). Let τ be the hitting time when $X_t = 0$ or $X_t = 1$ for the first time Find $f(x) = E_{x,0}[\tau]$. Assume $a \neq 0$ and $0 \le x \le 1$. Write the differential equation f satisfies and its boundary conditions. Find the solution. (This takes some calculation, possibly more than an actual exam question.)

10. Suppose X_t is standard Brownian motion. Find the value function

$$f(x,t) = \mathrm{E}_{x,t}\left[e^{\int_t^T W_s ds} \right] .$$

Write the backward equation that f satisfies. Show that this has a solution of the form $f(x,t) = e^{A(t)x+B(t)}$. Write the differential equations that A, and B satisfy and the values $A(T)$ and $B(T)$. Use these differential equations to find formulas for $A(t)$ and $B(t)$.

11. Consider the two equation model

$$\begin{aligned} dX_t &= M_t X_t dt + \sigma_X dW_{1,t} \\ dM_t &= -\gamma X_t dt + \sigma_M dW_{2,t} \ . \end{aligned}$$

The initial conditions are $X_0 = 0$ and $M_0 = 0$. Suppose that W_t and W_t are independent Brownian motions. Explain the backward equation approach to calculating $V(T) = \mathrm{E}[X_T^2]$. Define a value function. Give the backward equation and final conditions for this value function. Explain how to use this value function to evaluate $V(t)$.

12. Suppose X_t is a three dimensional Brownian motion with $|X_0| > r$. Suppose $\tau = \min\{t \text{ so that } |X_t| = r\}$.

    (a) Show that $|X_t|^{-1}$ is a martingale, if $|X_t| > 0$.

    (b) Show that
    $$\Pr(\tau < \infty) = \lim_{n\to\infty} \Pr(\tau < n) \ .$$

    (c) Show that, for all n, $\Pr(\tau \leq n) \leq \frac{r}{|X_0|}$. State the martingale theorem involved and explain how it applies.