Assignment 7, due November 4

Corrections: (none yet)

Theory questions:

1. (*Poisson simulation*) This is a long list of exercises related to the Poisson process leading to a simulation algorithm for continuous time Markov chains.

   (a) Show that an exponential random variable has the following Markov type property, for any $s > 0$ and $t > 0$:

   $$\mathrm{P}(T \in [t+s, t+s+ds] \mid T > t) = \mathrm{P}(T \in [s, s+ds]) \ .$$

   You interpret this as follows: If T is the time something breaks, if it has not broken by time t, then at time t it is "as good as new" in its initial state. Do this using the probability density $T \sim f(t) = \lambda e^{-\lambda t}$. Write out the formula for the left side using Bayes' rule for conditional probability, and show that the result is equal to the left side, which is $\lambda e^{-\lambda s} ds$.

   (b) Let $f_n(t_1, \ldots, t_n)$ be the joint density of the first n arrivels $T_1, \ldots, T_n$. Show that

   $$f_n(t_1, \ldots, t_n) = \begin{cases} \lambda^n e^{-\lambda t_n} & \text{if } 0 < t_1 < t_2 \cdots < t_n \\ 0 & \text{otherwise.} \end{cases}$$

   (c) The sets of ordered $n-$tuples in a cube of side length t is $A_n(t) \subseteq \mathbb{R}^n = \{t_1 < t_2 < \cdots < t_n < t\}$. Show that $\mathrm{vol}(A_n) = \frac{1}{n!}t^n$. Hint: There are several ways to do this. Choose just one way. (1.) The cases $n = 1$ and $n = 2$ are "obvious". Do the general case by induction. Use an integral expression for $A_n(t)$ as an integral of $A_{n-1}(t_n)$ over t_n. (2.) The volume of the whole cube is t^n. Suppose π is a permutation of the integers $1, \ldots, n$, and define $A_{n,\pi} = \{t_{\pi_1} < \cdots < t_{\pi_n}\}$. For example, if π puts the numbers in the order $(2, 1, 3, \ldots, n)$, then $A_{n,\pi} = \{t_2 < t_1 < t_3 \cdots < t_n\}$. All the sets $A_{n,\pi}$ have the same volume, and the number of such sets is the number of permutations, which is $n!$.

   (d) Show that the PDF of T_n (the $n-th$ arrival time) is

   $$f_n(t_n) = \lambda \left(\lambda t_n\right)^{n-1} \frac{e^{-\lambda t_n}}{(n-1)!} \ .$$

   (e) Show that if the number of arrivals in the interval $[0, a]$ is n, then these n arrival times are uniformly distributed and independent in $[0, a]$, modulo order. This means, you can generate the n arrival times by creating n independent uniforms in $[0, a]$ then sorting them.

   (f) Use these ideas to complete the demonstration that you can generate a Poisson process in $[0, T]$ by first choosing N_t and then choosing N_t arrival times independently.

(g) Consider the discrete time approximation of the Poisson process that puts a *hit* in the time interval $[t_j, t_{j+1}]$ with probability $\lambda\Delta t$, and otherwise leaves the interval empty. Suppose all hitting events are independent. This means $\mathrm{P}\left(N^{\Delta t}_{t_{j+1}} = N^{\Delta t}_{t_j}\right) = 1 - \lambda\Delta t$, and $\mathrm{P}\left(N^{\Delta t}_{t_{j+1}} = N^{\Delta t}_{t_j} + 1\right) = \lambda\Delta t$. Let T_1 be the first arrival time for the Poisson process and $T^{\Delta t}_1 = \min\left\{t_j \text{ with } N^{\Delta t}_{t_j} = 1\right\}$ be the first arrival time for the discrete time approximate Poisson process. Show that the discrete time probability "density" converges to the true probability density for T_1, which is $f(t) = \lambda e^{-\lambda t}$, by showing that if t_j is the largest discrete time less than t, then

$$\mathrm{P}\left(T^{\Delta t}_1 = t_j\right) = f(t)\Delta t + O(\Delta t^2) \ .$$

In this limit, $t_j \to t$ and $j \to \infty$ as $\Delta t \to 0$.

(h) Consider a combination of independent Poisson processes with rates λ_1 and λ_2 respectively. Let $N_t = N_{1,t} + N_{2,t}$ be the counting function that counts arrivals from both processes. Show that N_t is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. Hint: use the characterization from part (g).

2. (*Stationary states for a linear Gaussian process*) Consider the linear Gaussian process in continuous time $dX_t = -\gamma X_t dt + \sigma dW_t$. Let $u(x,t)$ be the probability density of X_t. The discrete time approximation is $X^{\Delta t}_t$, which is defined by

$$X^{\Delta t}_{t_{j+1}} = X^{\Delta t}_{t_j} - \gamma X^{\Delta t}_{t_j}\Delta t + \sigma\sqrt{\Delta t} Z_j \ ,$$

Where $Z_j \sim \mathcal{N}(0,1)$, i.i.d. Let $u_j(x)$ be the probability density of $X^{\Delta t}_{t_j}$. The mean and variance are $\mu_j = \mathrm{E}\left[X^{\Delta t}_{t_j}\right]$ and $\tau^2_j = \mathrm{var}\left(X^{\Delta t}_{t_j}\right)$.

(a) Show by induction on j that $u_j(x)$ is a Gaussian density for all j.

(b) Find a formula for $\mu_j = \mathrm{E}\left[X^{\Delta t}_{t_j}\right]$.

(c) Find a recurrence relation for τ^2_j, which is a formula for τ^2_{j+1} in terms of τ^2_j.

(d) Use this recurrence relation, and the limit $\Delta t \to 0$ to find a differential equation for $v(t) = \mathrm{var}(X_t)$.

(e) Write the solution to this differential equation, and use it to find the formula for $u(x,t)$.

(f) Use similar reasoning to show for the damped driven harmonic oscillator from class that

$$\frac{d}{dt}\mathrm{E}\left[\omega_0^2 X_t^2 + V_t^2\right] = 2\gamma \mathrm{E}\left[V_t^2\right] + \sigma^2 \ . \tag{1}$$

Next week we will do this quickly using Ito's lemma.

3. Consider the derivation of the linear noisy oscillator model in the notes, but use a different noise model. Assume that the noise in the velocity is directly proportional to the speed. This mean that the infinitesimal variance is proportional to the square of the speed. Write the SDE for this model. You will have to invent your notation for the constants in the new model. Derive the analogue of (1) for this model using the reasoning of problem 2.

Computing: Download and run `Assignment7.R`. Out of the box you should get:

```
> source("Assignment7.R")
Hello
got mean =  0.6947326 , and variance =  0.1081604 , and error bar =  0.001040002
```

The mean is an estimate of $f(0,0)$ for some final time and payout (see code for details). The error in the mean has two sources, the *statistical error* and the *bias*. The *error bar* reported above is an estimate of the standard deviation of the mean. If you used 100 times more samples, this should decrease by about a factor of 10. As you let the number of samples go to infinity, this mean does not converge to $f(0,0)$ because $\Delta t > 0$ in the SDE time step approximation. The bias is the amount by which the finite Δt mean differs from $f(0,0)$. We reduce Δt to reduce the bias.

1. Use the PDE solver from last week to compute the true value of $f(0,0)$. Give computational evidence that the Monte Carlo computation of this week converges to this answer as $N_s \to \infty$ (the number of samples) and $\Delta t \to 0$.

2. Copy the Monte Carlo code in `Assignment7.R` into the code from `Assignment6.R` to get a code that can evaluate f both ways and plot the results in the same figure. For the parameters of `Assignment7.R`, evaluate $f(x,0)$ at $N_p = 40$ uniformly spaced x values in $[-a,a]$. Choose the values of N_s and Δt that make the finite difference and Monte Carlo calculations agree to within "plotting accuracy" – which is the size of the symbol in the plot. Please do not use giant symbols that make this too easy. Note: the Δt for the finite difference calculation is determined by Δx and cannot be tuned independently. The two time step parameters (for Monte Carlo and finite differences) will be different. You know that the finite difference calculation can be made quite accurate with small enough Δx, so take the finite difference computation with pretty small Δx as the exact answer. Comment on the relative times it takes the two computational methods to run.

3. Repeat the experiment from part 2 with a sawtooth final condition. Plot the finite difference and Monte Carlo results in the same figure. The Monte Carlo method comes out looking better here because its accuracy does not depend on smoothness of the final condition.

4. Repeat the experiment from part 3 using one interesting drift value from Assignment 6. You will have to modify the Monte Carlo path generator to add in a drift.