Assignment 5, due October 21

Corrections: (none yet)

1. (*A generalization of the Ito isometry formula*) Here is a handy to calculate some things about Ito integrals

   (a) Suppose f_t and g_t are non-anticipating functions, and the corresponding Ito integrals are

   $$X_t = \int_0^t f_s dW_s$$
   $$Y_t = \int_0^t g_s dW_s \; .$$

   Show that

   $$\mathrm{cov}(X_t, Y_t) = \mathrm{E}[X_t Y_t] = \int_0^t \mathrm{E}[f_s g_s] \, ds \; .$$

   Notice that if $g_t = f_t$, this becomes the formula from the Week 5 notes. The argument can be similar to the argument for the Ito isometry formula in the notes.

   (b) Suppose $f_t = t^2$ and $g_t = 1$. The notes for Week 5 show that $X_t \sim \mathcal{N}(0, t^5/5)$. Clearly $Y_t = W_t$. Compute the covariance of X_t and W_t using the result of part (a).

   (c) Since (X_t, W_t) is a bivariate normal whose variance/covariance structure you know, you can compute the conditional variance $\mathrm{var}(X_t|W_t)$. Use this result to show that X_t is not a function of W_t. This is an example of a general phenomenon, that the value of an Ito integral depends on the whole path $W_{[0,t]}$, not just the endpoint W_t.

2. (*quadratic variation*) Let X_t be a stochastic process of the form

   $$X_t = \int_0^t f_s \, dW_s \; . \tag{1}$$

   The *quadratic variation* is $[X]_t$ defined by

   $$[X]_t = \lim_{m\to\infty} \sum_{t_j < t} \left(X_{t_{j+1}} - X_{t_j} \right)^2 \; . \tag{2}$$

This formula uses the definitions of the Week 5 notes. In particular, $\Delta t = 2^{-m}$ and $t_j = j\Delta t$. This exercise is a sequence of steps leading to the formula

$$[X]_t = \int_0^t f_s^2 ds \ . \tag{3}$$

This formula, like Ito's lemma next week, is a version of $(dW)^2 = dt$. The Ito integral (1) may be written informally as $dX_t = f_t dW_t$. Slightly more formally, if Δt is small, then $\Delta X = X_{t+\Delta t} - X_t \approx f_t \Delta W_t$. A term on the sum on the right in (2) is $(\Delta X_{t_j})^2 \approx f_{t_j}^2 (\Delta W_j)^2$. The informal derivation of (3) is

$$[X]_t = \int_0^t (dX_s)^2 = \int_0^t f_s^2 (dW_s)^2 = \int_0^t f_s^2 ds \ .$$

If you take the differential of both sides, you see that $(dX_t)^2 = f_s^2 (dW_t)^2 = f_s^2 dt$. The theorem related to this stuff is that: (i) the limit (2) exists, and (ii) it satisfies the integral formula (3). During this exercise, assume that for $s > t$,

$$\mathrm{E}\Big[(f_s - f_t)^2 \mid \mathcal{F}_t \Big] \le C(s-t) \ . \tag{4}$$

This hypothesis was used to show the convergence of Riemann sums to the Ito integral.

This exercise is a long sequence of steps of the kind in this week's material. Quadratic variation is important, but this exercise is here largely to demonstrate different ways to use the ideas. We assume that $\mathrm{E}\big[f_t^2\big] \le F_2$ and $\mathrm{E}\big[f_t^4\big] \le F_4$ for all t. Assume that f_t is a continuous function of t (unnecessary in the one place it is used).

(a) Show that if X_t is not given by (1), but is a smooth (differentiable, or twice differentiable, say) function of t, then the definition (2) gives $[X]_t = 0$.

(b) Prove the following variant of the Borel Cantelli lemma. Suppose A_m is a sequence of random variables. If

$$\sum_{m=1}^{\infty} \mathrm{E}[|A_m|] < \infty \ , \tag{5}$$

then $A_m \to 0$ as $m \to \infty$. Then show that $A_m \to 0$ as $m \to \infty$ under the hypothesis that

$$\sum_{m=1}^{\infty} \mathrm{E}\big[A_m^2\big] < \infty \ . \tag{6}$$

Hint for the second part: define $B_m = A_m^2$ and use the first part for B_m.

(c) The following notations simplify the calculations below. "Bin" number j is $B_j = [t_j, t_{j+1}]$. A term in the quadratic variation approximation sum is

$$\left(X_{t_{j+1}} - X_{t_j}\right)^2 = \left(\int_{B_j} f_t \, dW_t\right)^2 .$$

For $t \in B_j$ we make the approximation $f_t \approx f_{t_j}$. Show, in this notation, that

$$\int_{B_j} f_t \, dW_t = U_j + V_j \ ,$$

with

$$U_j = f_{t_j} \Delta W_j \ , \quad \text{and} \quad V_j = \int_{B_j} \left(f_t - f_{t_j}\right) \, dW_t \ .$$

This easy step and the next just establish notation.

(d) Show that

$$\sum_{t_j < t} \left(X_{t_{j+1}} - X_{t_j}\right)^2 = S_1 + S_2 + S_3 \ ,$$

where

$$\begin{aligned} S_1 &= \sum_{t_j < t} f_{t_j}^2 \Delta W_j^2 \\ S_2 &= 2 \sum_{t_j < t} U_j V_j \\ S_3 &= \sum_{t_j < t} V_j^2 \ . \end{aligned} \tag{7}$$

(e) Show that $S_{2,m} \to 0$ as $m \to \infty$. We write $S_{2,m}$ instead of our usual S_2 only in formulas that involve a sum over m. Hint: First

$$\mathrm{E}[S_2] \leq \sum_{t_j < t} \mathrm{E}[|U_j| \, |V_j|] \ ,$$

then Cauchy Schwarz, then $\mathrm{E}[U_j] \leq F_2 \Delta t$, then

$$\mathrm{E}\left[V_j^2\right] \leq C \int_0^{\Delta t} s \, ds \ .$$

The last step uses the Ito isometry formula and (4). You learn from this that $\mathrm{E}[|U_j| \, |V_j|] \leq C \Delta t^{3/2}$.

(f) Show that $S_{3,m} \to 0$ as $m \to \infty$.

(g) Write $S_1 = S_4 + S_5$, where

$$S_4 = \sum_{t_j < t} f_{t_j}^2 \Delta t$$
$$S_5 = \sum_{t_j < t} f_{t_j}^2 \left(\Delta W_j^2 - \Delta t \right) .$$

Show that

$$S_4 \to \int_0^t f_s^2 \, ds \ , \qquad \text{as} \quad m \to \infty .$$

Hint: this is a theorem of ordinary calculus.

(h) Show that $S_{5,m} \to 0$ as $m \to \infty$. Hint: compute $\mathrm{E}\left[S_5^2\right]$. Show that the off diagonal terms vanish. Show that the diagonal terms are bounded by $2F_4 \Delta t^2$.

3. (*Extra credit, do this only if you have time and energy*) Show that

$$\int_0^t e^{W_s} \, dW_s = e^{W_t - \frac{1}{2}t} - 1 .$$

This will be an easy calculation once we have proven Ito's lemma. But this asks you do do it directly from the definition. Write

$$e^{W_t - \frac{1}{2}t} - 1 = \sum_{t_j < t} \left(e^{W_{t_{j+1}} - \frac{1}{2}t_{j+1}} - e^{W_{t_j} - \frac{1}{2}t_j} \right)$$
$$= \sum_{t_j < t} e^{W_{t_j} - \frac{1}{2}t_j} \left(e^{\Delta W_j - \frac{1}{2}\Delta t} - 1 \right) .$$

Now do a Taylor expansion, recognize part as the Riemann sum approximation to the Ito integral. The hard part is controlling the remainder from the Taylor expansion. I give no hint except to say it's not exactly like the class demonstrations.