Sample questions for the final exam.

Instructions for the final:

- The final is Monday, December 16 from 7:10 to 9pm
- Explain all answers, possibly briefly. A correct answer with no explanation may receive no credit.
- You will get 20% credit for not answering a question. Points will be subtracted if you give a wrong answer.
- Cross off anything you think is wrong. You will have points subtracted for wrong answers even if the correct answer also appears.
- You may use one $8\frac{1}{2} \times 11$ piece of paper, a *cheat sheet* with anything you want written on it.
- The actual exam will be shorter than this.

True/False. In each case, state whether the statement is true of false and give a few words or sentence of explanation.

1. If X_t is a stochastic process with $\mathrm{E}\Big[(X_{t+\Delta t} - X_t)^2 \mid \mathcal{F}_t\Big] = a_t \Delta t + O(\Delta t^2)$ and $\mathrm{E}\Big[(X_{t+\Delta t} - X_t)^4 \mid \mathcal{F}_t\Big] = o(\Delta t)$, then $\mathrm{E}\Big[|X_{t+\Delta t} - X_t|^3 \mid \mathcal{F}_t\Big] = o(\Delta t)$.

2. If $dX_t = W_t^2 dt$, then
$$\lim_{\Delta t \to 0} \sum_{t_k < t} \left(X_{t_{k+1}} - X_{t_k}\right)^2 = 0 \; .$$
This uses our usual notation: $\Delta t > 0$ is the time step, $t_k = k\Delta t$ is a discrete time, $t > 0$ is a fixed time as $\Delta t \to 0$.

3. If X_t is a diffusion process and a martingale, and if a_t is adapted, then
$$Y_t = \int_0^t a_s dX_s$$
is a martingale and a diffusion process.

4. Suppose P and Q are probability measures on a measure space Ω. If P is absolutely continuous with respect to Q, then Q is absolutely continuous with respect to P.

5. If X_t^m is a family of random variables with probability density $u^m(x,t)$, and $u^m(x,t) \to u(x,t)$ as $m \to \infty$, then $X_t^m \to X_t$ as $m \to \infty$ almost surely.

Multiple choice In each case only one of the possible answers is correct. Identify the correct answer and explain why.

6. Suppose P and Q are equivalent probability measures on a measure space Ω. Which of the following is true

(a) If $A \subseteq \Omega$ is an event, then $P(A) = 0$ if and only if $Q(A) = 0$.

(b) There is a function $f(x,t)$ so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.

(c) $f(x,t_1) = E_{x,t_1}[f(X_{t_2}, t_2)]$, where $f(x,t)$ is the function so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.

(d) $E[V(X_T) \mid \mathcal{F}_t] = E[V(X_T) \mid \mathcal{G}_t]$

7. Suppose

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= -kX_t - \gamma V_t + \sigma dW_t \end{aligned}$$

Which of the following is true?

(a) X_t is a one dimensional martingale.

(b) X_t is a one dimensional Markov process.

(c) (X_t, V_t) is a two dimensional Markov process.

(d) $\mathrm{E}\Big[\big(X_t - \overline{X}_t\big)^4\Big] = 4\,\mathrm{var}(X_t)^2$. Here $\overline{X}_t = \mathrm{E}[X_t]$.

8. Suppose

$$\begin{aligned} dX_t &= X_t Y_t dt + dW_{1,t} \\ dY_t &= (X_t + Y_t)dt + dW_{2,t} \end{aligned}$$

and $(X_0, Y_0) = (1,1)$. Here, $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions. Which PDE below is satisfied by the value function

$$f(x,y,t) = \mathrm{E}_{x,y,t}\left[\exp\left(-\int_t^T X_t^2 dt\right)\right] .$$

(a) $\partial_t f + xy\partial_x f + (x+y)\partial_y f + \frac{1}{2} \triangle f - x^2 = 0$.

(b) $\partial_t f + xy\partial_x f + (x+y)\partial_y f + \frac{1}{2} \triangle f + x^2 f = 0$.

(c) $\partial_t f + \partial_x (xyf) + \partial_y ((x+y)f) + \frac{1}{2} \triangle f - x^2 f = 0$.

(d) $\partial_t f + xy\partial_x f + (x+y)\partial_y f + \frac{1}{2} \triangle f - x^2 f = 0$.

9. Consider the integration-by-parts formula

$$\int_0^t sX_s dX_s = \frac{1}{2}\left(tX_t^2 - t^2\right) - \frac{1}{2}\int_0^t \left(X_s^2 - s\right) ds .$$

This formula is true under which hypotheses:

(a) Only when X_t is a standard Brownian motion

(b) As long as X_t is an Ito process with $dX_t = a_t dt + dW_t$

(c) As long as X_t is an Ito process

(d) Only if X_t is a differentiable function of t.

Standard long answer questions

10. Suppose $X_{n+1} = 2X_n - X_{n-1} + Z_n$, where $Z_n \sim \mathcal{N}(0,1)$ i.i.d. Describe a value function and backward equation that can be used to calculate $\mathrm{P}(X_T > 1 \mid X_0 = X_1 = 0)$.

11. An order book holds *orders*, which are offers to buy or sell a given stock at a given price. An order in the book can be removed, or *cancelled* at any time. Suppose new orders arrive to the order book as a Poisson process with rate parameter λ. Suppose that the time until cancellation of any order is exponential with rate constant μ. Suppose all cancellations and inter-arrival times are independent. An *event* in the order book is an arrival of a new order or a cancellation of an existing order.

    (a) Suppose there are n orders at the book at time t. What is probability distribution of the time until the next event?

    (b) What is the probability that the next event is a cancellation?

    (c) Assume there is a statistical steady state (there is). What is the expected number of orders in this steady state?

12. Suppose σ_n is a sequence of numbers with

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty \; .$$

Suppose that $X_n \sim \mathcal{N}(\sigma^2)$. Do not assume that the random variables X_n are independent of each other. Show that

$$\sum_{n=1}^{\infty} X_n^2 < \infty \; .$$

13. Suppose $0 < T_1 < T_2 < \cdots$ are the random arrival times of a Poisson arrival process with rate constant λ. We want to construct a re-weighting function $L(t_1, t_2, \ldots)$ so that if $F(t_1, t_2, \ldots)$ is a function of the arrival times, then

$$\mathrm{E}_\mu[F(T_1, T_2, \ldots)] = \mathrm{E}_\lambda[F(T_1, T_2, \ldots) L(T_1, T_2, \ldots)] \; .$$

We want to simulate a Poisson arrival process with rate λ, compute the functional F, reweight with the likelihood ratio L, and get the expectation with respect to the Poisson process with rate constant $\mu \neq \lambda$. The formula for L depends on the details.

    (a) Find a formula for L if we simulate the λ process up to a fixed time t. The number of arrivals up to time t is uncertain. Hint: Imagine that you simulate the Poisson process by first choosing the number of arrivals and then choosing the arrival times independently and uniformly in $[0, t]$, then sort them.

(b) Find the formula for L if we simulate the λ process until a fixed number n of arrivals have happened. Hint: the joint PDF of the T_k depends only on T_n and the ordering.

14. Suppose X_n is a discrete time Gaussian process $X_{n+1} = aX_n + bZ_n$, where $Z_n \sim \mathcal{N}(0,1)$ are independent for different n and $X_0 = 0$. We are interested in the hitting probability

$$A = \mathrm{P}(|X_n| \geq M \text{ for some } n \leq T) \ .$$

Describe a Monte Carlo algorithm that evaluates A by simulating the Brownian random walk $X_{n+1} = X_n + bZ_n$. This is the original process with $a = 1$. Give formulas for any quantities you need to compute, particularly likelihood ratios. Assume there is a random number generator that creates i.i.d. standard normals so that you can just use Z_n in formulas.

15. Let X_t be a Brownian motion with $\mathrm{E}\left[(X_t - X_0)^2\right] = t$, starting from $X_0 = a > 0$. Let τ be the hitting time $\tau = \min\{t \mid X_t = 0\}$. Calculate $\mathrm{E}\left[\tau^{-1}\right]$. Write the probability density or distribution function then calculate the appropriate integral involving it. Hint: use the change of variable $t = s^{-2}$ to do the integral.

16. Consider the square root process $dX_t = -aX_t dt + \sigma\sqrt{X_t} dW_t$. Suppose $X_0 = 1$.

(a) Use the Ito calculus to calculate $\frac{d}{dt}\mathrm{E}[X_t]$ and $\frac{d}{dt}\mathrm{E}\left[X_t^2\right]$. Use these results to write a formula for $\mathrm{E}[X_t]$ and a formula for $\mathrm{E}\left[X_t^2\right]$. Hint: if $\dot{v} = -2av + \sigma^2 e^{-at}$, you can find $v(t)$ using the ansatz $v = \alpha e^{-at} + \beta e^{-2at}$. You find α from the ODE and β from the initial condition, once α is known.

(b) Write the backward equation and use it to calculate $m(x,t) = \mathrm{E}_{x,t}[X_T]$. Assume an ansatz of the form $m(x,t) = A(t)x + B(t)$. Show that your answer is consistent with part (a).

(c) Use the backward equation to write a formula for $f(x,t) = \mathrm{E}_{x,t}\left[X_T^2\right]$. Assume an ansatz of the form $f(x,t) = A(t)x^2 + B(t)x + C(t)$.

(d) Use some of these results to show that

$$X_t \xrightarrow{\mathcal{D}} 0 \ , \quad \text{as } t \to \infty \ .$$