Assignment 7.

Due March 8.
Revised March 5 to fix question 2

1. Let T be a standard exponential random variable with density function $u(t) = e^{-t}$ for $t > 0$ and $u = 0$ for $t < 0$.

   (a) Show that $E[T^n] = n!$. Hint: $e^{-t} = -\partial_t e^{-t}$, now integrate by parts and use induction on n.

   (b) Calculate the characteristic function of T. Although the answer involves complex numbers, you do not need the theory of functions of a complex variable, but only algebra and ordinary calculus.

   (c) Show (informally, without worrying about convergence) that for any probability density $u(t)$,
   $$E[T^n] = i^n \partial_\xi^n \widehat{u}(\xi)\Big|_{\xi=0} \quad .$$
   This is the n^{th} derivative of $\widehat{u}$ evaluated at $\xi = 0$.

   (d) Verify by direct differentiation of the characteristic function formula from part (b) that the formula in part (a) is correct.

2. Use the Dirac formula (15) in lecture 4 to show that
$$(2\pi)^n \int_{x \in R^n} u(x)v(x)dx = \int_{\xi \in R^n} \overline{\widehat{u}(\xi)}\widehat{v}(\xi)d\xi \; .$$
Here, $\overline{\widehat{u}(\xi)}$ is the complex conjugate of $\widehat{u}(\xi)$. We assume u and v are real. This is the *Plancharel* formula. Verify this formula by calculating both sides explicitly when $u(x) = e^{-a(x-b)^2}$ and $v(x) = e^{-x^2/2}$. This is partly an exercise in computing characteristic functions of Gaussians.

3. Let X be an n component multivariate normal with covariance matrix C and $H = C^{-1}$. Define $Y = (X_1, \ldots, X_m)^*$ and $Z = (X_{m+1}, \ldots, X_n)^*$, so that $X = \begin{pmatrix} Y \\ Z \end{pmatrix}$. Naturally, $m < n$. The matrix H may be written correspondingly: $H = \begin{pmatrix} H_{YY} & H_{YZ} \\ H_{YZ}^* & H_{ZZ} \end{pmatrix}$, so that $x^*Hx = y^*H_{YY}y + 2y^*H_{YZ}z + z^*H_{ZZ}z$ (with y corresponding to Y, etc.). Show that the conditional distribution of Y given that $Z = z$ is an m dimensional multivariate normal with a certain mean and covariance matrix. Find expressions for this conditional mean and covariance in terms of z, H_{YZ}, and H_{YY}. In particular, suppose $n = 2$, $m = 1$, $\text{var}(Y) = \text{var}(Z) = 1$ and $\text{cov}(Y, Z) = a < 1$. What is the conditional distribution of Y given that $Z = z$? Hint: If the probability density for X is $u(y, z)$, then the conditional density for Y given that $Z = z$ is $\mathit{Const}(z)u(y, z)$. In the second formula, z plays the role of a parameter rather than a random variable.

4. This question explores some seeming paradoxes related to multivariate normals. Don't worry too much about the "How do you ..." parts. The point is to see that certain intuitions about Gaussians can be wrong.

   (a) Suppose $L = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$, $Z = (Z_1, Z_2)^*$ with Z_1 and Z_2 independent standard normals and $X = LZ$. If $a = b = 1/\sqrt{2}$, find the covariance matix of X. The formulas $X_1 = Z_1$ and $X_2 = aZ_1 + bZ_2$ seem to imply that X_1 is different from X_2 in that information used to create X_1 is reused in computing X_2, but there is no influence the other way. Show that the distribution of X_1, X_2 is the same as the distribution of X_2, X_1. How do you reconcile the symmetry of X_1 and X_2 with the difference in the way they were made?

   (b) Consider the 3×3 matrix

$$C = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & 1 \end{pmatrix} ,$$

   Show that this matrix is the covariance matrix of a 3 component multivariate normal, $(X, Y, Z)^*$, if $a^2 + b^2 \leq 1$. Suppose $a > 0$ and $b > 0$. Show that X being large positive influences Z to be large positive and Z being large positive influences Y to be large positive (which would make you think X influences Y), but that in fact X is independent of Y. How do you explain this?

5. A standard normal random walk of length n is an n component random variable, $X = (X_1, \ldots X_n)^*$, defined by $X_{n+1} = X_n + Z_n$ together with initial conditions $X_0 = 0$, when the $Z_k \sim \mathcal{N}(0, 1)$ are iid.

   (a) Write the probability density function $u(x)$ using the fact that conditional on $X_1, \ldots, X_k$, X_{k+1} is a normal with variance 1 and mean X_k. Then use Bayes' rule to turn this into a formula for the joint density.

   (b) Show that the expression in the exponent from part (a) may be written $x_1^2 + (x_2 - x_1)^2 + \cdots + (x_n - x_{n-1})^2 = x^* H x$. Identify the tridiagonal matrix H.

   (c) Use the fact that if $k > j$, then

$$X_k = X_j + \sum \text{ some standard normals}$$

   to calculate $C_{jk} = \text{cov}(X_j, X_k)$.

   (d) Check directly from these formulas that the answers to parts (b) and (c) satisfy $HC = I$. If it's too cumbersome to do for general n, check the case $n = 4$.