Assignment 9, last assignment.

Given December 5, due December 19 Last revised December 5.

Focus: Multidimensional diffusions and PDE

1. The Ornstein Uhlenbeck process is described by the pair of SDE's:

$$dV_t = -\gamma V_t dt + \alpha dW_t \ , \tag{1}$$

$$dX_t = V_t dt \ . \tag{2}$$

As originally proposed by Einstein and refined by Ornstein and Uhlenbeck, this is a model for the motion of a small particle in a fluid. Equation (1) describes the speed of the particle, V_t. It says that there is a random forcing, αdW_t, caused by collisions from molecules of the fluid, and a drag, $-\gamma V_t$, caused by friction with the fluid. Equation (2) just says that the rate of change of the position is given by the speed.

a. The equation (1) may be solved using the method of integrating factors. Show that if we multiply both sides by the integrating factor $e^{\gamma t}$, we get

$$d(e^{\gamma t} V_t) = \alpha e^{\gamma t} dW_t \ ,$$

in the Ito sense.

b. Use this to derive a formula

$$V_T = e^{-\gamma T} V_0 + \int_0^T m(T,t) dW_t \ ,$$

with a specific formula for m.

c. Use this to show that $V_t \sim \mathcal{N}(\mu_t, \sigma_t^2)$, and find formulae for μ_t and σ_t.

d. Suppose $u(v,t)$ is the PDF for V_t, write the forward equation PDE satisfied by u. This may be expressed in the form $\partial_t u = \mathcal{L}^* u$, where $\mathcal{L}$ is the generator of the process. (By convention, the generator is the operator that goes in the backward equation, so it's adjoint is what goes in the forward equation.)

e. From the answer to part c, together with the formula for the density of a $\mathcal{N}(\mu, \sigma^2)$ random variable, find a formula for $u(v,t)$.

f. Verify by explicit differentiation that this formula satisfies the forward equation from part d.

g. Find the limits

$$\mu_0 = \lim_{t\to\infty} \mu_t \ , \quad \text{and} \quad \sigma_0 = \lim_{t\to\infty} \sigma_t \ .$$

Show that $u(v)$, the $\mathcal{N}(\mu_0, \sigma_0^2)$ density, satisfies $\mathcal{L}^* u = 0$. This is the steady state velocity distribution. The relationship between α and γ for a given σ_0 is a special case of what physicists call the fluctuation dissipation theorem.

h. Express X_T as an integral involving W_t for $t \leq T$.

i. Use the integral expressions for X_t and V_t to show that (X_t, V_t) is a two dimensional normal and calculate the mean and variance-covariance matrix.

j. Use this to write a formula for $u(x, v, t)$, the joint PDF for (X_t, V_t). Notice that this is a smooth function of x and v for $t > 0$ even though (1)(2) represents a degenerate diffusion. Not all degenerate diffusions have this property.

k. Write the forward equation PDE for $u(x, v, t)$.

l. Verify that the formula from part j satisfies this PDE.

m. Show that for large t the variance of X_t has the approximate form $\text{var}(X_t) \approx D^2 t$. Find a formula for D in terms of γ and α. This formula, and its physical interpretation, was a significant part fo the research that won Einstein the Nobel Prize.