Assignment 8.

Given November 7, due November 14 Last revised November 8.

Focus: Ito calculus

1. We will calculate the first four Picard iterates for the SDE $dX_t = X_t dW_t$, with $X_0 = 1$. These are given by $X_t^{(0)} = 1$ for all $t \geq 0$, and

$$X_t^{(k+1)} = 1 + \int_0^t X_s^{(k)} dW_s \ .$$

This is partly to provide practice using the Ito calculus on the various stochastic integrals that come up.

a. Compute $X_t^{(1)}$.

b. Compute $X_t^{(2)}$. It will be simpler to compute

$$Y_t^{(2)} = X_t^{(2)} - X_t^{(1)} = \int_0^t (X_s^{(1)} - X_s^{(0)}) dW_s = \int_0^t Y_s^{(1)} dW_s \ .$$

c. The exact solution of the SDE is $X_t = e^{W_t} e^{-t/2}$. Expand e^{W_t} and $e^{-t/2}$ is Taylor series in W_t and t respectively, keeping terms up to and including $O(t)$. Multiply these to get a short time approximation to X_t up to $O(t)$. Show that this agrees with your answer to part b.

d. Calculate $Y_t^{(3)} = \int_0^t Y_s^{(2)} dW_s$. To simplify the answer, you need a relationship between $\int_0^t W_s ds$ and $\int_0^t s dW_s$.

e. Calculate $Y_t^{(4)} = \int_0^t Y_s^{(3)} dW_s$. Combine with previous results to get $X_t^{(4)}$.

f. Extend the calculation of part c to get an approximation containing all terms up to and including $O(t^2)$. Check that this agrees with your answer to part e.

2. a. Using our bounds for $X^{(k+1)} - X^k$ from Lecture 7, paragraph 1.5 with the constant in (3), show that $X_t - X_t^{(k)} = O(t^{(k+1)/2})$. For this you will need to know that

$$\sum_{j=k}^{\infty} z^j = \frac{z^k}{1-z} = O(z^k) \quad , \quad \text{as } z \to 0.$$

b. Justify the approximation $X_{t+\Delta t} = X_t + \sigma(X_t)\Delta W_{\Delta t} + O(\Delta t)$. Here, $\Delta W_{\Delta t} = W_{t+\Delta t} - W_t$. This uses part a and facts about Ito integrals. Hint: it might be easier notationally if you replace t by 0 and Δt by t.

c. Justify the approximation

$$X_{t+\Delta t} = X_t + \sigma(X_t,t)\Delta W_{\Delta t} + a(X_t,t)\Delta t + \frac{1}{2}\sigma(X,t)\partial_X\sigma(X_t,t)(\Delta W_{\Delta t}^2 - t) + O(t^{3/2}) \ .$$

This requires the approximation in part b and the technique behind it.

3. Suppose $dX_t = a(X_t,t)dt + \sigma(X_t,t)dW_t$ and $f(x,t) = E_{x,t}[V(X_T)]$. Use the fact that $f_t = f(X_t,t)$ is a martingale and Ito's lemma for solutions of SDEs to find a partial differential equation $\partial_t f + \cdots = 0$. This is another in our collection of useful backward equations.

4. Derive the Black Scholes formula for European option pricing. For this, we need the cumulative distribution of a standard normal random variable. This is, for $Z \sim \mathcal{N}(0,1)$,

$$N(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{z} e^{-y^2/2} dy \ .$$

Many integrals involving gaussians can be expressed in terms of N. For example, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $P(X \le \mu + z\sigma) = N(z)$. In earlier assignments, we have used the fact that $E[e^X]$ can be calculated by writing the integral, completing the square in the exponent, and recognizing this as another gaussian integral.

a. A geometric Brownian motion satisfies the SDE $dS_t = rSdt + \sigma S_t dW_t$. Note that what we usually call σ is now σS_t. Compute Y_t, the ordinary calculus guess at the solution (if the solution if W_t were a differentiable function of t). Use the Ito calculus to find and verify the correct solution in the form $S_t = A(t)Y_t$.

b. Find an expression for S_T in terms of S_t and $W_T - W_t$. This is just a line of algebra.

c. For $f(S,t) = E[V(S_T) \mid \mathcal{F}_t]$, write the backward equation for f. This asks you to translate the general result in question 3 into a more specific equation that holds for geometric Brownian motion. The resulting equation is called the "Black Scholes" equation.

d. For any final payout function $V(s)$, write a formula for $f(S,t)$ as the expected value of some function of a gaussian random variable whose mean and variance depend on t in a simple way. This is an application of the result of part b and the definition of f. It does not use the backward equation.

e. For $V(s) = \max(s-K,0) = (s-K)_+$, compute the integral in part d explicitly in terms of the N function. This is the "Black Scholes formula".

f. Verify by direct differentiation that the formula satisfies the backward equation.

5. Use Ito's lemma to show that each of the following is a martingale. Comment on the difficulty of doing it this way or directly, as in an earlier assignment.

a. $X_t = e^{k^2t/2}\sin(kW_t)$

b. $Y_t = W_t^4 - 6\int_0^t W_s^2 ds.$