Assignment 3.

Given September 20, due September 26.
Last revised, September 23.

Objective: Conditioning and Markov chains, II.

1. Suppose an $n \times n$ matrix, A, has n linearly independent right eigenvectors (column vectors) with corresponding eigenvalues λ_j: $Ar_j = \lambda_j r_j$. Suppose the the corresponding left (row vector) eigenvectors are L_j: $l_j A = \lambda_j l_j$. Suppose that the r_j and l_k have been normalized to be biorthogonal: $l_j r_k = \delta_{jk}$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$ (this is the "Kronecker delta"; the δ_{jk} are the entries in the identity matrix.).

- **a.** Show that $A = \sum_{j=1}^{n} \lambda_j r_j l_j$. Note that each term in the sum is an $n \times n$ matrix. In the other order, $l_j r_j$ is a 1×1 matrix.
- **b.** Find a similar expression for A^t, the product of A with itself t times, not the transpose of A.

2. We have a three state Markov chain wih transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} .$$

For example, some of the transition probabilities are $P(1 \to 1) = \frac{1}{2}$, $P(3 \to 1) = \frac{1}{3}$, and $P(1 \to 2) = \frac{1}{4}$. Let $\tau = \min(t \mid X_t = 3)$. Even though this τ is not bounded (it could be arbitrarily large), we will see that $P(\tau > t) \leq Ca^t$ for some $a < 1$ so that the probability of large τ is very small. This is enough to prevent the stopping time paradox (take my word for it). Suppose that at time $t = 1$, all states are equally likely.

- **a.** Consider the quantities $u_t(j) = P(X_t = j \text{ and } \tau > t)$. Find a matrix evolution equation for a two component vector made from the $u_t(j)$ and a submatrix of P.
- **b.** Solve this equation using the method of problem 1 to find a formula for $m_t = P(\tau = t)$. For this you will have to find the eigenvalues and the left and right eigenvectors of the 2×2 matrix A.
- **c.** Use the answer of part b to find $E[\tau]$. It might be helpful to use the formula

$$\sum_{t=1}^{\infty} tP(\tau = t) = \sum_{t=1}^{\infty} P(\tau \geq t) .$$

  Verify the formula if you want to use it.
- **d.** Consider the quantities $f_t(j) = P(\tau \geq t \mid X_1 = j)$. Find a matrix recurrence for them.

e. Use the method of question 1 to solve this and find the f_t.

3. Let P be the transition matrix for an n state Markov chain. Let $v(k)$ be a function of the state $k \in \mathcal{S}$. For this problem, suppose that paths in the Markov chain start at time $t = 0$ rather than $t = 1$, so that $X = (X_0, X_1, \ldots)$. For any complex number, z, with $|z| < 1$, consider the sum

$$E\left[\sum_{t=0}^{\infty} z^t v(X_t) \mid \mathcal{F}_0\right] . \tag{1}$$

Of course, this is a function of $X_0 = k$, which we call $f(k)$. Find a linear matrix equation for the quantities f that involves P, z, and v. Hint: the sum

$$E\left[\sum_{t=1}^{\infty} z^t v(X_t) \mid \mathcal{F}_1\right] .$$

may be related to (1) if we take out the common factor, z.

4. A particular random walk has states X_t that are integers, possibly negative. In general the transition probabilities are $P(k \to k-1) = P(k \to k) = P(k \to k+1) = \frac{1}{3}$ with all other transion probabilities being zero. Don't worry that the state space might be infinite or that stopping times might be infinite.

a. Show that $F_t = X_t$ and $G_t = X_t^2 - \frac{2}{3}t$ are martingales.

b. Define the stopping time $\tau_a = \min(t \mid |X_t| = a)$. Use the result of part a and the Doob stopping time theorem to show that $E(\tau_a \mid X_0 = 0) = \frac{3}{2}a^2$.

c. Let $f(k)$ be defined by $f(X_0) = E[\tau_a \mid \mathcal{F}_0]$. Find a system of linear equations satisfied by the numbers $f(k)$ for $-a < k < a$. Express these equations in the form $Af = \mathbf{1}$, where A is a $(2a-1) \times (2a-1)$ matrix and $\mathbf{1}$ is a vector of all ones. To find the solution, calculate Aw where $w(k) = 1$, $w(k) = k$, and $w(k) = k^2$, then express the answer as a linear combination of these vectors.

d. Show that the martingale method of part b also applies to $E[\tau_a \mid X_0 = k]$ for any $|k| < a$. Check that this is consistent with the answer to part c.