Assignment 2.

Given January 31, due February 7.

Objective: To work with Taylor series in elementary numerical analysis.

1. Examine curve A and curve B. One of the functions has a jump in its third derivative while the other is completely smooth. Determine which curve has a discontinuous third derivative and locate the discontinuity on the graph. Do not spend lots of time doing this in a quantitative way. The point is to show that a jump in the third derivative is hard to spot or pinpoint on a graph.

2. Extrapolation means estimating the value of a function from known values on one side only. For example, suppose $g(x)$ is a smooth function of x and we know the values $g(0)$, $g(h)$, $g(2h)$, and so on. We wish to estimate, say, $g(-h)$ or $g(-2h)$ from the known values. One instance would be the estimate (this is our most ambitious one)

$$g(-2h) \approx a_0 g(0) + a_1 g(h) + a_2 g(2h) + a_3 g(3h) \; . \qquad (1)$$

The coefficients may or may not depend on h. We can also construct estimates of $g(-h)$ and/or use different numbers of points. Show that we get first order accurate estimates of $g(-h)$ or $g(-2h)$ using only $g(0)$, second order accuracy with $g(0)$, and $g(h)$, and so on to fourth order accuracy with (1). Show that these extrapolations are exact for constants, linears, quadratics, and cubics respectively.

3. The interval $(0, L)$ is divided into n subintervals, called "cells", of equal length $\Delta x = L/n$. Use the notation $x_k = k\Delta x$ for the endpoints of the intervals, and $I_k = (x_k, x_{k+1})$ for the intervals themselves. It will be convenient to write $x_{k+1/2} = (k + 1/2)\Delta x$ for the midpoint of I_k. We do not have the values of f, but we do have the "cell averages"

$$F_k = \frac{1}{\Delta x} \int_{x_k}^{x_{k+1}} f(x)dx \; .$$

Assume that $f(x)$ is a smooth function. Show that $F_k = f(x_{k+1/2}) + O(\Delta x^2)$. Hint: it will be easier if you use a variable $y = x - x_{k+1/2}$ and perform Taylor series expansions about $x = x_{k+1/2}$, $y = 0$.

4. Find a second order accurate estimate of $f(x_k)$ in terms of the numbers F. The estimate will be different for the "interior" points, x_1, ..., x_{n-1} and the "boundary" points x_0 and x_n. For the interior points, find an approximation

$$f(x_k) = aF_{k-1} + bF_k + O(\Delta x^2) \; .$$

. The estimate of $f(0)$ should use F_0 and F_1.

5. Suppose we extrapolate a value for F_{-1} in terms of the given values F_0, F_1, etc. as in part 2, then estimate $f(0)$ using the interior estimate from part 4 and the extrapolated F values. What order of extrapolation do you need to get a second order estimate of $f(0)$?

6. Find a fourth order estimate of $f(x_{k+1/2})$ using F values. First find the interior formula, then figure out what order of extrapolation and how many extrapolated values you need to achieve fourth order at the end points.

7. Use the computer to demonstrate the properties of the method from part 6, first on the function $g(x) = sin(x^2)$, then on the function

$$f(x) = \begin{cases} \sin(x^2) & \text{if } x < r, \\ \sin(x^2) - (x-r)^3 & \text{if } x \geq r, \end{cases}$$

with $L = 2$, $r = \sqrt{2}$, and various values of n. To do this you must write a procedure that calculates F values exactly (in exact arithmetic). To show that the method works, you should be able to divide the error in the estimate of $f(x_{k+1/2})$ by Δx^4 and get a limit as $\Delta x \to 0$. This will be dicey in floating point, even in double precision. For this exercise, you may do all the programming in Matlab if you wish.

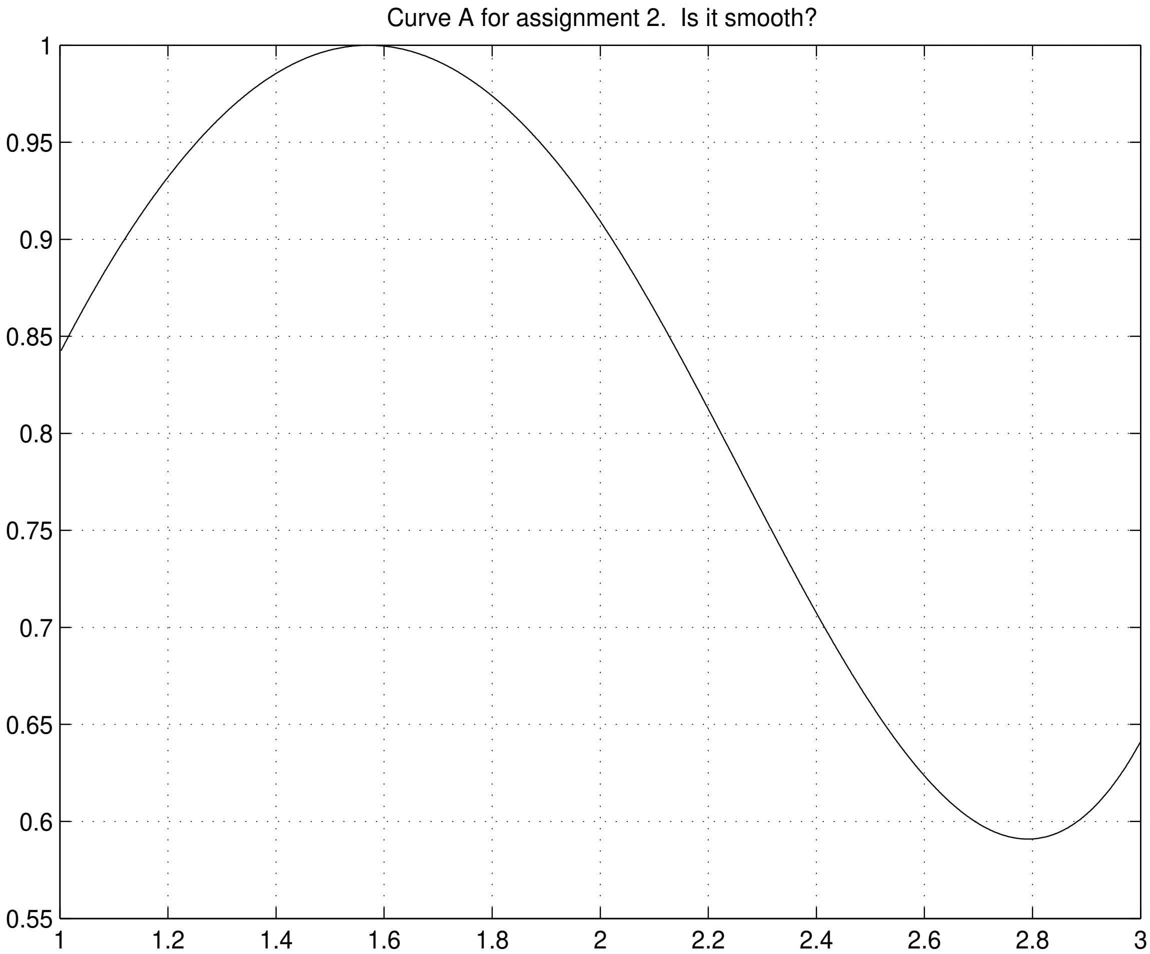

Figure 1: Figure A for question 1. Is this a smooth curve?

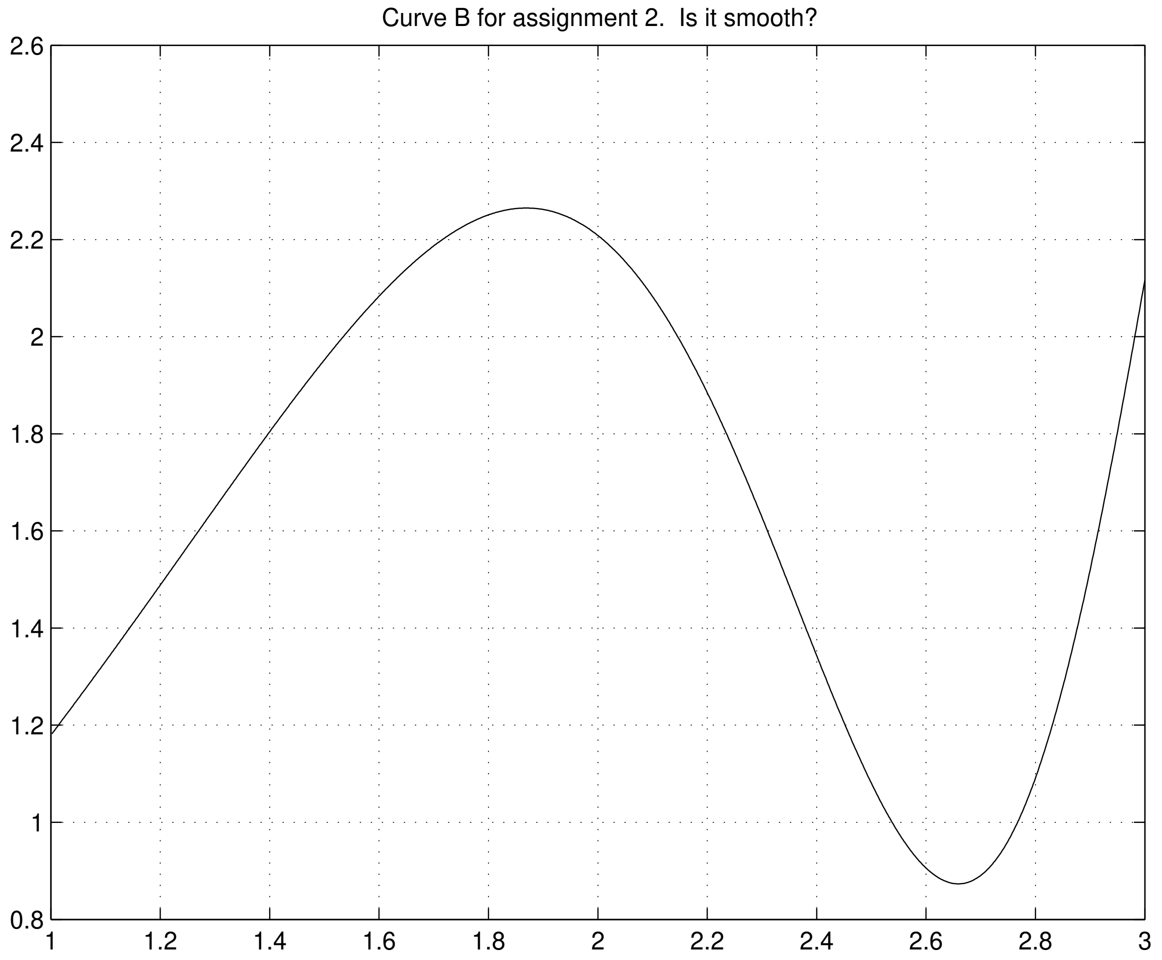

Figure 2: Figure B for question 1. Is this a smooth curve?