Numerical Methods II, Courant Institute, Spring 2024
http://www.math.nyu.edu/faculty/goodman/teaching/NumericalMethodsII2024.html
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Assignment 5, Fourier stability analysis, boundary layers

This assignment concerns a linear advection diffusion equation in one space dimension and time

$$\partial_t u + a\partial_x u = \mu \partial_x^2 u + S(x) \ . \qquad (1)$$

The source function $S(x)$ is specified and the goal is to find the corresponding solution $u(x,t)$. In computational examples, take

$$S(x) = 1 \ , \quad 0 < x < L \ .$$

We normalize the advection velocity to be one (by non-dimensionalizion if necessary):

$$a = 1 \ .$$

Use Dirichlet boundary conditions

$$u(0,t) = 0 \ , \quad u(L,t) = 0 \ .$$

Initial data are that u starts out being identically zero

$$u(x,0) = 0 \ , \quad 0 < x < L \ .$$

You may take $L = 1$ in computational examples, again by non-dimensionalization if necessary. The diffusion coefficient μ will vary. We are particularly interested in the solution when μ is small.

Take a uniform mesh with n internal grid points $x_j = j\Delta x$ and

$$\Delta x = \frac{L}{n+1} \ .$$

Suppose V is a grid function with values V_j for $1 \leq j \leq n$. The formulas below implicitly use the ghost cell values $V_0 = 0$ and $V_{n+1} = 0$. We define the centered first and second order difference operators as

$$(D_0 V)_j = \frac{1}{2\Delta x}\left(V_{j+1} - V_{j-1} \right)$$
$$(D_+ D_- V)_j = \frac{1}{\Delta x^2}\left(V_{j+1} - 2V_j + V_{j-1} \right)$$

A *Fourier mode* is a grid function $V_j(\theta) = e^{i\theta j}$. Be aware that Fourier modes do not satisfy the given Dirichlet boundary conditions. If M is any translation invariant operator, the corresponding symbol $m(\theta)$ is defined by

$$MV(\theta) = m(\theta)V(\theta) \ .$$

A semi-discrete approximation to (1) involves a time-dependent grid function $U_j(t)$

$$\dot{U} = -D_0 U + \mu D_+ D_- U + S \; . \tag{2}$$

Exercise 1. Suppose that S is the grid function $S_j = S(x_j)$ and $u(x,t)$ is a smooth function of[1] x and t, and $U_j(0) = u(x_j, 0)$. Compute the symbol of the right side of (2). In other words, define the translation invariant operator $M = -D_0 + \mu D_+ D_-$ and compute $m(\theta)$. Use the result to show the semi-discrete scheme is von-Neumann stable. Show that the computed solution is second order accurate in the sense that, for any fixed t,

$$\Delta x \sum_{j=1}^{n} \left[\, u(x_j,t) = U_j(t) \,\right]^2 = O(\Delta x^2) \; .$$

Use forward Euler to make a fully discrete scheme. Define the time k grid vector $U_j^{(k)} \approx U_j(t_k)$, with $t_k = k\Delta t$. The scheme is

$$U^{(k+1)} = U^{(k)} + \Delta t \left[\, M U^{(k)} + S \, \right] \; . \tag{3}$$

Without the advection term, the scheme would be stable under the CFL condition

$$\mu \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \; . \tag{4}$$

Exercise 2. Code the forward Euler scheme (3). Suppose μ is small (say $\mu = .1$ or $\mu = .01$). Do numerical experiments (plot the numerical solution) on a variety of meshes up to time $t = \frac{1}{2}$ and several values of Δx to see whether the put diffusion CFL condition (4) determines the stability. Make plots of $|m(\theta)|$ for $0 \leq \theta \leq 2\pi$ to see whether the numerical stability/instability you see in the code agrees with the stability prediction based on $\max |m(\theta)|$.

Exercise 3. Use good Δx and Δt values from Exercise 2 to make a movie of an accurate simulation of (1) up to the time the solution seems to have settled into a steady state. Describe the time dependent and steady state behavior when μ is small. There will be a *boundary layer* either near $x = 0$ or near $x = L$. Most of the behavior may be explained using the simplification $\mu = 0$, but not the boundary layer.

Exercise 4. Consider the partly implicit scheme

$$U^{(k+1)} = U^{(k)} + \Delta t \left[\, D_0 U^{(k)} + \mu D_+ D_- U^{(k+1)} \, \right] \; . \tag{5}$$

Show that the scheme is first order accurate if $\Delta t = \Delta x$ and $\mu = 1$. This means showing the residual has the appropriate size and the scheme is stable.

[1] This is easy to show, assuming the source function is smooth and has the right boundary behavior, using techniques from PDE. If you haven't taken a PDE theory class, you can infer this smoothness from computations below.

Exercise 5. Code a tri-diagonal solver to find V that satisfies

$$(I - cD_+D_-)\, V = F \; .$$

Use this to implement the partly implicit scheme (5). See whether you can do the computation of Exercise 3 faster, first for $\mu = 1$ and then for smaller values of μ.

Exercise 6. Try to find time stepping scheme using D_0 and D_+D_- that is second order accurate and stable with an "advective" time step $\Delta t = \lambda \Delta x$. One possibility is a split scheme that does the advective part using an explicit four state Runge Kutta method and the diffusive part using Crank Nicholson. Does this work (is it stable and second order)?