1 Finite differences for elliptic PDE

Big example: $\Delta u = f$ in square

$S = \square = [0,1] \times [0,1]$, $u = 0$ on ∂S.

Grid points $(x_i, y_j) = (ih, jh)$

$(n+1)h = 1$

<u>interior</u> points $x_1, \dots, x_n$, $y_1, \dots, y_n$

$N = n^2$ of them

boundary points $x_0 = 0$, $x_{n+1} = 1$, $y_0 = 0$, $y_{n+1} = 1$.

$u_{ij} \cong u(x_i, y_j)$, $f_{ij} = f(x_i, y_j)$

Consistent approximation

$$\Delta u = \partial_x^2 u + \partial_y^2 u$$

$$\cong \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$

$$+ \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

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Discrete approx:

(*) $\frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right) = f_{ij}$

$$\Delta_h u = f$$

Discrete Laplace operator, 5-pt Laplacian

large system of linear eqns: $N = n^2$.

sophisticated solution methods

Accuracy: $\|u - u_{exact}\|_{\infty} \leq c\, h^2$ ← 2nd order accurate

in many senses.

Consistency let $\bar{u}_{ij} = u_{exact}(x_i, y_j)$

Then $\Delta_h \bar{u} - f = R = O(h^2)$

Pf: $\Delta_h \bar{u}_{ij} = \frac{\left[u(x_i+h, y_j) + u(x_i-h, y_j) + \cdots\right]}{h^2}$

$= \Delta u(x_i, y_j) + h^2 ??(D^4\bar{u})$

$= f(x_i, y_j) + h^2$ "

③

so

$$(\Delta_h \bar{u} - f)_{ij} = R_{ij} = h^2 (\ast\ast)$$
$$\le c \cdot h^2 \quad \text{all } i,j$$

if u is C^4.

<u>Stability:</u> $\Delta_h (u - \bar{u}) = R$

$$\Longrightarrow \quad \| u - \bar{u} \|_{\ast} \le c \cdot \| R \|_{\ast\ast} = O(h^2).$$

This is an <u>estimate</u> (inequality), which depends on a <u>norm</u>.

<u>Will do 3</u>

$L^2 - \ell^2$, energy estimate	$L^\infty = \ell^\infty$ maximum principle	L^1, ℓ^1 dual to maximum principle.

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Stability via energy estimates.

<u>step 1</u> prove the equations are solvable:

Δ_h has no kernel, $\Delta_h v = 0 \Rightarrow v = 0$.

PDE version $\Delta v = 0$ in S, $v = 0$ on ∂S

$\Rightarrow v = 0$ inside S.

<u>Pf</u> $\Delta v = 0 \Rightarrow \iint_S v\left(\partial_x^2 v + \partial_y^2 v\right) dx\,dy = 0$

$$\Rightarrow \iint_S |\nabla v|^2\, dx\,dy = 0$$

$\Rightarrow \nabla v \equiv 0$ inside S, so $v = 0$ too.

Discrete version $v = \{v_{ij}\} \in \mathbb{R}^N = \mathbb{R}^{n^2}$.

$$\|v\|_{\ell^2}^2 = h^2 \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij}^2 \quad \left(\approx \iint v^2\, dx\,dy\right)$$

inner product $\langle v, w\rangle_{\ell^2} = h^2 \sum_{ij} v_{ij} w_{ij}$

if $\Delta_h v = 0$ then $v = 0$

$$h^2 \sum_{ij} v_{ij} \frac{v_{ij+1} - 2v_{ij} + v_{ij-1}}{h^2} + h^2 \sum_{ij} v_{ij} \frac{v_{i+1j} - 2v_{ij} + v_{i-1j}}{h^2}$$

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$$= -h^2 \sum_{i,j} \left[\left(\frac{v_{i+1,j} - v_{i,j}}{h} \right)^2 + \left(\frac{v_{i,j+1} - v_{i,j}}{h} \right)^2 \right]$$

$$= 0$$

This implies $v_{i+1,j} = v_{i,j}$ and

$v_{i,j+1} = v_{i,j}$ for ~~all~~ all i,j

so $v_{i,j} = 0$ because $v_{i,j} = 0$ for $i=0$ or $j=0$ or $i=n+1$ or $j=n+1$.

Step 2 quantitative version:

Poincare inequality ~~closure of~~

PDE version: if v is C_2 in $\bar{S}$ and and $v=0$ on ∂S then

$$\iint_S v^2(x,y)\,dx\,dy \le C \cdot \iint_S |\nabla v(x,y)|^2\,dx\,dy.$$

Find the constant by

$$\min_{v \neq 0} \frac{\iint |\nabla v|^2\,dx\,dy}{\iint_S v^2\,dx\,dy} = \text{Rayleigh quotient.}$$

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General Rayleigh Quotient

$$\min_{v \neq 0} \frac{\langle v, Av \rangle}{\langle v, v \rangle}$$ if A is symmetric

and positive definite is ~~given in~~ determined by the smallest eigenvalue and eigenvector

$$A v = \lambda_{min} v$$

$$\min \frac{\langle v, Av \rangle}{\langle v, v \rangle} = \lambda_{min}$$

Here, as before $\iint |\nabla v|^2 = -\int v \cdot \Delta v$

$$= \langle v, \Delta v \rangle_{L^2(S)}$$

So we want eigenvalues & eigenfunctions

$-\Delta v = \lambda v$, ans

$$v(x,y) = \sin(\pi x) \cdot \sin(\pi y)$$

$$\lambda = 2\pi^2 \sim 20$$

<u>Discrete version</u>: $-\Delta_h v = \lambda v$,

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~~$v = \sin(\pi x) \sin$~~

$$v_{ij} = \sin(\pi x_i) \sin(\pi y_j)$$

$$\lambda = 2 \cdot 2 \cdot \frac{1 - \cos(\pi h)}{h^2}$$

$$\simeq 2\pi^2 \quad \text{(as before)}$$

Therefore, in the square

if $\Delta_h v = R$, then

$$\|v\|_{\ell^2} \lesssim \frac{1}{2\pi^2} \cdot \|R\|_{\ell^2}$$

Therefore $\|u - \bar{u}\|_{\ell^2} \le c \cdot h^2$

<u>Higher order methods</u> - 2 ways

<u>standard way</u>, use a more accurate Δ_h

<u>mehrstellenverfahren</u> (sp?) (also called "compact scheme" - use a smarter right hand side.

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~~Higher~~ More accurate ~~top~~ discrete laplacian.

Richardson Extrapolation

$g(x)$ smooth fn of x

$$D_h^2 g = \frac{1}{h^2}\left(g(x+h) - 2g(x) + g(x-h)\right)$$

$$= g_{xx} + \frac{1}{12} h^2 g_{xxxx} + O(h^4)$$

$$D_{2h}^2 g = \frac{1}{(2h)^2}\left(g(x+2h) - 2g(x) + g(x-2h)\right)$$

$$= \cancel{g_{xx}}\ g_{xx} + \frac{1}{12}(2h)^2 g_{xxxx} + O(h^4)$$

Take a linear combination to eliminate the principal error term

$$4D_h^2 g - D_{2h}^2 g = 3g_{xx} + O(h^4)$$

$$g_{xx} = \frac{4}{3} \cdot \frac{1}{h^2}\left(g(x+h) - 2g(x) + g(x-h)\right)$$

$$- \frac{1}{12}\frac{1}{h^2}\left(g(x+2h) - 2g(x) + g(x-2h)\right)$$

$$g_{xx}(x_i) \simeq \frac{1}{h^2}\left\{-\frac{1}{12}g_{i+2} + \frac{4}{3}g_{i+1} - \frac{5}{2}g_i + \frac{4}{3}g_{i-1} - \frac{1}{12}g_{i-2}\right\} + O(h^4)$$

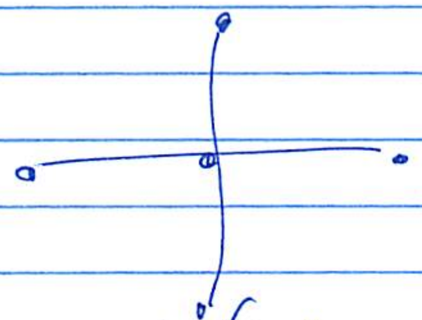
(In obvious notation)

$$\frac{1}{h^2}\left(-\frac{1}{12}u_{i+2,j}+\cdots\right)=f_{ij}$$

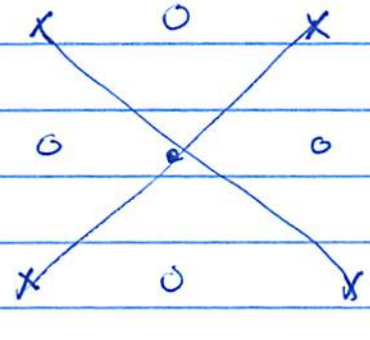
if it is stable (it is), the result is 4^{th} order.

<u>Drawback:</u> bigger stencil requires more boundary conditions.

<u>Compact scheme</u>



$$\Delta_h u = \Delta u + h^2\cdot\frac{1}{12}\left(\partial_x^4 u+\partial_y^4 u\right)+O(h^4)$$



rotated laplace operator

$$\frac{1}{2}\frac{1}{h^2}\left(u_{i+1,j+1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i-1,j-1}-4u_{ij}\right)$$
$$=\Delta u+h^2 R+O(h^4)$$

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$$\cancel{R = \left(\frac{1}{\sqrt{2}}(\partial_x + \partial_y)\right)^4 u}$$

$$R = 2h^2 \cdot \frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}(\partial_x + \partial_y)\right)^4 u + \left(\frac{1}{\sqrt{2}}(\partial_x - \partial_y)u\right)^4\right]$$

$=$

lots of algebra

$$\frac{2}{3}\Delta_h u + \frac{1}{3}\Delta_h^{\times} u = f + \frac{1}{12}h^2 \Delta f$$

4th order (cf Stable).

Disadvantage: lots of ~~algra~~ algebra, hard to apply to more complicated PDE.