1 Numerical Methods II 1/26/2012

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Class one,

Time stepping solution of ODE

$$\dot{x} = f(x) \qquad x = (x_1, \cdots, x_d) \in R^n$$

Start with $x_0 = x(0)$,

approximate $x(t)$, $t > 0$

Time step $= \Delta t$

$t_n = n \Delta t$ (non-uniform time steps also possible)

x_n = numerical approx $\cong x(t_n)$

Euler method - forward, explicit

$$x_{n+1} = x_n + f(x_n) \Delta t$$

Error bound: assume $\|f(x)\| \le M$

$\|f'(x)\| \le M$ (Jacobian matrix, norm of)

Thm: Then there is a $C(t)$

so that

$$\|x(t_n) - x_n\| \le \Delta t \, C(t_n)$$

Proof Consistency + stability argument

Consistency: The exact ODE solution is an approximate Euler solution

stability: approximate Euler solutions are close to exact Euler solutions.

Lemma (consistency): Let $y_n = x(t_n)$, then

$$y_{n+1} = y_n + \Delta t \, f(y_n) + \Delta t \, r_n$$

where

$$(*) \qquad \|r_n\| \le C \cdot \Delta t \, .$$

Comment: r_n is the residual, the amount by which the eqn is not satisfied.

(3) Some say the residual is $\Delta t\, r_n = \tilde{r}_n$ with $\|\tilde{r}_n\| \le C \cdot \Delta t^2$. Many of them call $\tilde{r}_n$ <u>local</u> <u>truncation</u> <u>error</u>.

The convention (*) makes the error, $x_n - y_n$, the same power of Δt as the residual. More generally, $\|r_n\| \le C \cdot \Delta t^p$ is accuracy of order p. Euler is first order accurate.

<u>Proof</u>: $$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(t))\,dt$$

$$= x(t_n) + \int_{t_n}^{t_{n+1}} f(x(t_n))\,dt + \int_{t_n}^{t_{n+1}} \big(f(x(t)) - f(x(t_n))\big)\,dt$$

This gives

$$r_n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \big(f(x(t)) - f(x(t_n))\big)\,dt.$$

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$\|r_n\| \le$ average of $\|f(x(t)) - f(x(t_n))\|$.

But

$$\|f(x(t)) - f(x(t_n))\| \le \max_x \|f'(x)\| \cdot \|x(t) - x(t_n)\|$$

By hypothesis, $\|f'(x)\| \le A$ for all x.

Since $\|\dot{x}\| = \|f(x)\| \le A$, $\cancel{x(t) - x(t_n)}$

$$\|x(t) - x(t_n)\| \le A \cdot (t - t_n) \le A \cdot \Delta t$$

Altogether

$$\|f(x(t)) - f(x(t_n))\| \le A^2 \Delta t,$$

which proves the consistency estimate ⊛.

<u>Lemma</u> (stability) if $|A \Delta t| \le \frac{1}{2}$,

$$x_{n+1} = x_n + \Delta t\, f(x_n)$$

$$y_{n+1} = y_n + \Delta t\, f(y_n) + \Delta t\, r_n$$

$$\|x_0 - y_0\| \le C_0 \Delta t$$

Then $\|x_n - y_n\| \le C(t_n)\, \Delta t$

(there is a $C(t)$ indep. of Δt so that ...)

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Proof:

$$\|x_{n+1} - y_{n+1}\| \le \|x_n - y_n\| + \Delta t \|f(x_n) - f(y_n)\| + \Delta t \|r_n\|$$

But $\|f(x_n) - f(y_n)\| \le A \cdot \|x_n - y_n\|$, so

$r < c\,\Delta t$

~~x_{n+1} x_n~~

(**) $$\|x_{n+1} - y_{n+1}\| \le (1 + A\Delta t)\|x_n - y_n\| + \Delta t\, r$$

where $r = \max \|r_n\|$

Take a "semigroup" approach to bounding ~~$\|x_{n+1} - y_{n+1}\|$~~ $\|x_n - y_n\|$ using this. Define

$S_{n,k}$ for $n \ge k$ by

$S_{k,k} = 1$ (initial condition)

$S_{n+1,k} = (1 + A\Delta t) S_{n,k}$ (dynamics)

"clearly" $S_{n,k} \le e^{A(t_n - t_k)}$.

Also

$$\|x_n - y_n\| \le S_{n,0}\|x_0 - y_0\| + \sum_{k<n} S_{n,k}\, \Delta t \|r_n\|$$

(prove by induction on n) $\le r\Delta t \sum_k S_{nk}$

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$$\|x_n - y_n\| \le e^{At_n}\|x_0 - y_0\| + c\Delta t \sum e^{A(t_n - t_m)}$$

$$\le e^{At_n}\|x_0-y_0\| + c\int_0^t e^{A(t_n-t)}dt$$

$$= C_0 \Delta t\, e^{At_n}$$

$$+ c\Delta t \frac{\left(e^{At_n}-1\right)}{A} \qquad \underline{\text{QED.}}$$

Remark (for later): The same argument applies if

$$x_{n+1} = B x_n + \Delta t\, f(x_n)$$

$$y_{n+1} = B y_n + \Delta t\, f(y_n) + \Delta t\, r_n$$

if $\|Bx\| \le \|x\|$ — B is a contraction.

Asymptotic Error Expansion:

$$\cancel{x_n = x(t_n) + \Delta t\, x_n^{(1)} + \Delta t^2 x_n^{(2)} + \cdots}$$

$$x_n = x(t_n) + \Delta t\, x^{(1)}(t_n) + \Delta t^2 x^{(2)}(t_n) + \cdots$$

where $x^1(t)$, $x^2(t)$, … are smooth functions of t.

⑦

Plug in & expand, collect terms, etc

$x_{n+1} = x_n + \Delta t\, f(x_n)$

$x(t_n + \Delta t) + \Delta t\, x^1(t_n + \Delta t) + \Delta t^2\, x^2(t_n + \Delta t) + \cdots$

$= x(t_n) + \Delta t\, x^1(t_n) + \Delta t^2 x^2(t_n) + \cdots$

$+ \Delta t\, f\left(x(t_n) + \Delta t\, x^1(t_n) + \cdots\right)$

$\Rightarrow \cancel{x(t_n)} + \underline{\dot{x}(t_n)\,\Delta t} + \underline{\underline{\tfrac{1}{2}\ddot{x}(t_n)\,\Delta t^2}}$

$\cancel{+\Delta t\, x^1(t_n)} + \underline{\underline{\dot{x}^1(t_n)\,\Delta t^2}} + \cdots$

$+ \Delta t^2 \cancel{x^2(t_n)}$

$= \cancel{x(t_n)} + \Delta t\, \cancel{x^1(t_n)} + \Delta t^2 \cancel{x^2(t_n)} + \cdots$

$+ \Delta t\left(\underline{f(x(t_n))} + \underline{\underline{f'(x(t_n))\,\Delta t\, x^1(t_n)}} + \cdots\right)$

Order Δt terms:

$\dot{x}(t_n) = f(x(t_n))$ (Duh!)

Order Δt^2 terms

$\tfrac{1}{2}\ddot{x}(t_n) + \dot{x}^1(t_n) = f'(x(t_n))\, x^1$

But $\dot{x} = f(x)$, so

$$\ddot{x} = f'(x)\,\dot{x} = f'(x)\,f(x) \quad \text{and}$$

$$\underline{\dot{x}^{(1)} = f'(x)\,x^{(1)} - \tfrac{1}{2} f'(x)\,f(x)}$$

principal error equation.

<u>Modified Equation approach</u>: Find the ode that x_n satisfies – the modified eqn.

Find $g(x) = f(x) + \delta t\, f_1(x) + \cdots$ modified eqn

so that if $\boxed{\dot{x} = g(x)}$ then

$$\cancel{x(t_{n+1}) = x(t_n) + \delta t\, g(x(t_n)) + O(\delta t^3)}$$

$$x(t_{n+1}) = x(t_n) + \delta t\, f(x(t_n)) + O(\delta t^3)$$

That is: $x_{n+1} = x_n + \delta t\, f(x_n)$ is a first order accurate approx. for $\dot{x} = f(x)$ but a 2^{nd} order approx for $\dot{x} = g(x)$.

Similar algebra:

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if $\dot{x} = g(x)$ then

$$\ddot{x} = g'(x)\, g(x) \quad \text{and}$$

$$x(t_{n+1}) = x(t_n) + \dot{x}(t_n)\,\Delta t + \tfrac{1}{2}\ddot{x}(t_n)\,\Delta t^2 + \cdots$$

$$= x(t_n) + g(x(t_n))\,\Delta t + \tfrac{1}{2} g'(x(t_n))\, g(x(t_n))\,\Delta t^2$$

$$+ \cdots$$

$$\cancel{x_{n+1} = x_n + g(x_n)\,\Delta t}$$

$$x(t_{n+1}) = x(t_n) + f\,\Delta t + f_1\,\Delta t^2 + \tfrac{1}{2} f' f\,\Delta t^2$$

$$+ O(\Delta t^3)$$

Therefore cancel the Δt^2 terms:

$$f_1 = -\tfrac{1}{2} f' f$$

Thus, doing

$$x_{n+1} = x_n + \Delta t\, f(x_n)$$

is like doing

$$\dot{x} = f(x) - \tfrac{1}{2} f'(x)\, f(x).$$

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Examplb: $\ddot{x} = -x$ — harmonic oscillator,
Formulate as 1^{st} order system

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = -x = -x_1$$

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$f(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \qquad f'(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$f'f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$

$$= \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

Modified eqn

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Matrix is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{2} & 1 \\ -1 & \frac{\Delta t}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with soln, e.g.

$$x_1(t) = \cos(t)\, e^{\frac{\Delta t}{2} \cdot t}$$

$\Rightarrow$ slow spiral out.