Monte Carlo Methods, Fall 2007, Courant Institute, NYU
Homework 5, Due December 34

1. Show that if a discrete time Markov chain satisfies detailed balance and $C(t)$ is the time t lag auto-covariance function in equilibrium, then $C(2t) \geq 0$ for all t.

2. We want to estimate

$$A = E\left[\int_0^1 X(t)dt\right] ,$$

over paths with $X(0) = 1$, $X(1) = 1$ and $dX = -X^2dt + XdW$.

   (a) Given $\Delta t = 1/n$, suggest a sensible discrete approximation to the integral defining A and the SDE. Use this to create an n dimensional integral that approximates A. Let $F(\vec{x})$ be the probability density in this integral.

   (b) Suggest and implement a dynamic Monte Carlo strategy for sampling F that uses single site updates. These single site updates can by done using Metropolis, but you need to design a careful trial distribution that either is symmetric itself or whose asymmetry you can understand for the rejection step.

   (c) Suggest and implement a dynamic Monte Carlo strategy for sampling F that uses a global Langevin algorithm of the kind discussed on the previous homework. The time step for Langevin part will be related to, but not the same as, $\Delta t = 1/n$. Adjust the Langevin time step so the acceptance probability is somewhere near $1/2$.

   (d) Use your code from Homework 4 to estimate the auto-correlation time and create error bars for both of the dynamic samplers. Try to tune the samplers to reduce τ. Use this τ and an estimate of the static variance (which does not depend on the dynamic sampling strategy) to create error bars for both estimators.

   (e) For each of the samplers, plot the auto-correlation time (estimated) as a function of n and try to guess (from the plots) how τ depends on n for large n.

   (f) Which estimator is better for small Δt? Be careful to compare the two approaches fairly by considering the amount of work per Monte Carlo step for each.

   (g) Check the correctness of of the codes and algorithms in parts (b) and (c) by checking that they are consistent with each other. Do this by checking that they give $\widehat{A}$ values that are within each other's error bars and by checking that $X(1/2)$ has the same distribution (compare the histograms). Do this for $n = 4$ and $n = 100$, just in case a bug in the sampler is hard to see for small or large n.