Mathematics of Finance, Courant Institute, Spring 2019
https://www.math.nyu.edu/faculty/goodman/teaching/MathFin2019/MathFinance.html

Compound interest

This is an upper level undergraduate applied math class. An important part of any such class is learning to use mathematical tools in modeling and estimation. Compound interest is a good place to start. The applied math tool is Taylor series approximations. Not the infinite sum, but the approximation from just one or two terms.

The book describes the value of a dollar after T years with interest rate r compounded m times per year as

$$V(m,r) = \left(1 + \frac{r}{m}\right)^{mT} \ .$$

The *continuous compounding* limit is the limit $m \to \infty$. The book mentions the theorem that

$$\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{mT} = e^{rT} \ .$$

One point of this theorem, for an applied mathematician doing finance, is that the exponential formula on the right is simpler and easier to use than the power formula on the left.

You should know when an approximation is applicable and how accurate it is. These are two ways of stating the same issue. You can use an approximation if it is accurate enough for your application. So, how close is $\left(1 + \frac{r}{m}\right)^{mT}$ to e^{rT} for some specific m? Here's a trick to find out: You're raising something $(\cdots)$ to a high power, which is mT. You can see what's happening by expressing it as the exponential of something. If $a > 0$ is any number, and $log(a)$ is the "natural log', the log base e, then

$$a = e^{\log(a)} \ .$$

Therefore

$$1 + \frac{r}{m} = e^{\log(1+\frac{r}{m})} \ .$$

When m is large, and if r is not too large, then $\frac{r}{m}$ is small. If $\frac{r}{m}$ were zero, then we would have $\log(1) = 0$. Since $\frac{r}{m}$ is small, $\log(1 + \frac{r}{m})$ should be close to zero.

We find out how close using a Taylor series approximation. Suppose x is small (close to zero). If $f(x)$ is a "good" function (most functions are good in this sense unless you know they're not), then

$$\begin{aligned} f(x) &\approx f(0) \\ f(x) &\approx f(0) + xf'(0) \\ f(x) &\approx f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) \\ &\text{etc.} \end{aligned}$$

The fancier approximations are more accurate, but we want to use the simplest one that suffices for our purpose.

We have $f(x) = \log(1+x)$, and we will take $x = \frac{r}{m}$. The derivatives are

$$
\begin{aligned}
& f(0) = 0 \\
f'(x) = \frac{1}{x} \quad , \quad & f'(0) = 1 \\
f''(x) = \frac{-1}{x^2} \quad , \quad & f''(0) = -1 \\
& \text{etc.}
\end{aligned}
$$

We write out the Taylor series approximations concretely, for $f = \log$ and $x = \frac{r}{m}$:

$$
\begin{aligned}
\log\left(1 + \frac{r}{m}\right) &\approx 0 \\
\log\left(1 + \frac{r}{m}\right) &\approx \frac{r}{m} \\
\log\left(1 + \frac{r}{m}\right) &\approx \frac{r}{m} - \frac{1}{2}\frac{r^2}{m^2} \\
&\text{etc.}
\end{aligned}
$$

If we use the first approximation, we get

$$\left(1 + \frac{r}{m}\right)^{mT} \approx e^{0 \cdot mT} = e^0 = 1 \ .$$

This isn't very useful for compound interest. The second approximation gives

$$\left(1 + \frac{r}{m}\right)^{mT} \approx e^{\frac{r}{m} \cdot mT} = e^{rT} \ .$$

This is more useful. It gives the exponential approximation to the compound interest formula. But it doesn't say how accurate this approximation is.

The third formula gives

$$
\begin{aligned}
\left(1 + \frac{r}{m}\right)^{mT} &\approx e^{\left(\frac{r}{m} - \frac{1}{2}\frac{r^2}{m^2}\right) \cdot mT} \\
&= e^{rT - \frac{1}{2}\frac{r^2 T}{m}} \\
&= e^{rT} e^{-\frac{1}{2}\frac{r^2 T}{m}} \ .
\end{aligned}
$$

This says that the continuous compounding exponential approximation differs from the actual compound interest formula by a factor of

$$e^{-\frac{1}{2}\frac{r^2 T}{m}} \ .$$

First, note that this is less than one because it's the exponential of something negative. It should be less than one, because the value increases when you compound more. If you compound infinitely, the value should be a little bigger than if you compound m times.

But how much less than one? We answer that question with another Taylor approximation, this time for the exponential function. We have

$$\begin{aligned} f(x) = e^x \quad &, \quad f(0) = 1 \\ f'(x) = e^x \quad &, \quad f'(0) = 1 \\ f''(x) = e^x \quad &, \quad f''(0) = 1 \\ & \qquad\qquad \text{etc.} \end{aligned}$$

This gives the sequence of approximations to the exponential function

$$\begin{aligned} e^x &\approx 1 \\ e^x &\approx 1 + x \\ e^x &\approx 1 + x + \frac{x^2}{2} \\ &\text{etc.} \end{aligned}$$

We apply these with $x = \frac{-r^2T}{2m}$. The first approximation gives no information. The second one is

$$e^{\frac{-r^2T}{2m}} \approx 1 - \frac{-r^2T}{2m} \ .$$

This says that $e^{\frac{-r^2T}{2m}}$ is less than one by about $\frac{-r^2T}{2m}$. That's what we need to know. Now we go back to our approximation earlier and put in this information. The result is

$$\left(1 + \frac{r}{m}\right)^{mT} \approx e^{rT}\left(1 - \frac{r^2T}{2m}\right) \ .$$

For example, suppose $r = .05$ (five percent interest per year) and $T = 10$ (ten years) and $m = 4$ (four compoundings per year). The exact value at the end of ten years is

$$\left(1 + \frac{.05}{4}\right)^{4\cdot 10} = 1.6436194634870103 \ .$$

The exponential approximation is

$$e^{.05\cdot 10} = e^{.5} = 1.6487212707001282 \ .$$

The exponential is higher than the exact amount by

$$1.6487212707001282 - 1.6436194634870103 = 0.005101807213117926$$

That seems accurate at first, but it is off by 51 *basis points* (a basis point is one precent of one percent), which can be a lot in some high-tech finance. Our Taylor approximation predicts that the exponential will be off by

$$e^{.05\cdot 10}\,\frac{(.05)^2 \cdot 10}{2 \cdot 4} = 0.005152253970937902 \ .$$

This is also 51 basis points.