Mathematics of Finance, Courant Institute, Spring 2019
https://www.math.nyu.edu/faculty/goodman/teaching/MathFin2019/MathFinance.html
Always check the class message board before doing any work on the assignment.

Assignment 2, due February 19

Corrections: (none yet)

Test your understanding. These questions involve only basic reasoning. Please answer each one with one or two simple grammatical sentences. These are worth two points each. One point is for the correct answer and one is for the correct and grammatical explanation.

1. Suppose $r > 0$ and $T > t$, then the forward price $F = e^{r(T-t)}S_t$ is larger than the spot price S_t. Does this imply that S_T (the spot price at time T, which is unknown at time t) will be larger than S_t? Does this imply that investors expect $S_T > S_t$?

2. Suppose an asset is traded with a known price only once per year. Suppose there is a forward contract to deliver the asset at time T, which is not one of the trading dates. Suppose $t = 0$ is a trading date, and the asset price then is S_0. Is it true that the value zero forward price should be $S_0 e^{rT}$.

3. The yield curve is *inverted* when
   (a) Interest rates for longer term Treasury bonds are listed before interest rates for short term Treasury notes.
   (b) The interest rate for some Treasury bonds is negative.
   (c) The interest amount you pay for a dollar in 20 years is less than the amount you pay for a dollar in 5 years.
   (d) The interest rate for longer term Treasury bonds is lower than the interest rates for short term Treasury notes.

4. Suppose there is an `R` function `fun(t)`. Suppose one of the commands in this function is `x = 4` What will these `R` commands produce?

```
> x = 10
> u = fun(x)
> x
```

   Explain your answer in terms of the environment where the commands in `fun(t)` are executed and the environment where the three lines above are executed.

From the book, pages 22 and 23, problems
1,
2,

3 (currencies are supposed to be driven by differences in interest rates between countries),
4,
6.
Also, 7: Suppose there is a contract to buy an asset for price K at time T. Suppose the market price of this contract at time $t < T$ is F_t. Find a formula for S_t (the asset spot price at time t) in terms of r, t, T, F_t, and K.

Computing

This exercise uses computation to look at *single life* annuities from the point of view of an insurance company that sells them. A single life annuity annuity makes fixed payments m times per year as long as the holder is alive. Each payment is $\frac{1}{m}C$. A typical single life annuity has $m = 12$, which means monthly payments. The payments end when the individual dies. When the insurance company sells a single life annuity, it don't know how long it will be paying. Actuaries and financial analysts deal with this.

A standard approach is to make a probability model of the time of death and use this to estimate the expected present value of the payment stream. Suppose f is the amount of a payment scheduled for time t in the future (from the sale date). The present value of f is fe^{-rt}. But the payment may not happen. The actual payment is F, which is equal to f (if the payment happens) or 0 (if the holder of the annuity dies first). A random variable with two values like this is called *Bernoulli*. Suppose that p is the probability that a payment is made. Then the expected payment is

$$\mathrm{E}[\,F] = p \cdot f + (1-p)\cdot 0 = pf \ .$$

The expected present value of the payment is $e^{-rt}\mathrm{E}[\,F] = e^{-rt}pf$.

Suppose p_n is the probability that the annuity holder is alive for the n^{th} payment. Then the expected present value of all the payments is

$$v = \sum_{n=1}^{\infty} e^{-rt_n} p_n \frac{1}{m} C \ .$$

The n^{th} payment time (measured from the sale date) is $t_n = \frac{n}{m}$. The payment amount $\frac{1}{m}C$ is a common factor. Therefore, the expected present value may be written as

$$v = \frac{C}{m}\sum_{n=1}^{\infty} e^{-\frac{rn}{m}} p_n \ . \tag{1}$$

The expected present value (liability if you're the insurance company) depends on the interest rate r and the survival probabilities p_n.

Actuaries model survival probabilities using the *force of mortality*, $\mu(a)$. Here, a is the age of the person $\mu(a)$ describes the probability that a person who is alive at age a will die soon. Let a_n be the age of the person at payment date t_n. The force of mortality is a model of the conditional probability of dying between t_n and t_{n+1} if the holder is alive at time t_n. That is:

$$\Pr(\text{ living at } t_{n+1} \mid \text{living at } t_n) \approx 1 - (t_{n+1} - t_n)\mu(a_n) \; .$$

The quantity $(t_{n+1}-t_n)\mu(a_n)$ is the probability of a person dying between times t_n and t_{n+1}. This probability is proportional to the force of mortality: larger μ means more death probability. It is also proportional to the time interval: twice the probability of dying in an interval twice as long. The formula has an "approximately equal" sign, $\approx$, because it becomes exact only in the limit $m \to \infty$. This is the same as the limit $\Delta t = t_{n+1} - t_n \to 0$. But $m = 12$ is large enough (we hope) that we can use this formula, with equality, to calculate probabilities.

On the right in this formula is the conditional probability. This is defined using *Bayes' rule*. If A and B are any events, then

$$\Pr(B|A) = \frac{\Pr(B \text{ and } A)}{\Pr(A)}$$

Apply this with A being the event that the holder is alive at time t_n and B being the probability that the holder is alive at time t_{n+1}. Then $\Pr(A) = \Pr(\text{ alive at } t_n) = p_n$, and $\Pr(B) = p_{n+1}$. Also $\Pr(B \text{ and } A) = p_{n+1}$ (think about this). These two formulas, and some algebra, leads to

$$p_{n+1} = (1 - (t_{n+1} - t_n))\, \mu(a_n)\, p_n \; . \tag{2}$$

This *recurrence relation* determines all the survival probabilities p_n. We know $p_0 = 1$, because t_0 is the time the annuity is sold to a living person. We then take $n = 0$ in (2) and find p_1. Knowing p_1, we take $n = 1$ and find p_2, and so on. In principle the numbers p_n are never zero. But with a reasonable model of the force of mortality, it should be extremely unlikely that a person lives to age 120 (the age of Moses).

1. Suppose the force of mortality is a constant, λ. Find a formula for p_n and then for v. This involves finding and then summing a geometric series. This model is unrealistic in that a 20 year old person has the same chance to live to 21 as a 100 year old person has to live to 101.

2. The constant force model of part (a) unrealistic. It implies that a 20 year old person has the same chance to live to 21 as a 100 year old person has to live to 101. More realistic mortality models have $\mu(a)$ increasing with a. A popular model is *Makeham's law*, which is

   $$\mu(a) = \lambda + \alpha e^{\beta a} \; .$$

   Reasonable parameter values seem to be $\lambda = .001$, $\beta = .085$, and $\alpha = .01$. There are no simple formulas for p_n for Makeham's law. Write an `R` script that contains a function `makeham(lam, al, bet, m, a0, max)` that creates and returns an array with n values $\mu(a_n)$ for $n = 1, \ldots, \max$. The parameters represent λ, α, β, m, and a_0. (It might be simpler not to use an array here, but please use one anyway. Part of the exercise is getting practice with an array.)

3. Write an R script called `sla.R` (for *single life annuity*) that prints a well formatted table of the present expected value of a single life annuity as a function of the age when the annuity starts. Do the ages in 5 year increments running from 40 to 80. For each age, use your `makeham()` function to get the survival probabilities (out to age 200, just to be safe?) and then add up the present expected values of the monthly payments. Assume an interest rate $r = 5\%$. When you are done, you should be able to type `source("sla.R") [enter]` at the command line in the console window and get the table.