Assignment 7, due November 18

Corrections: (none yet. See message board)

1. We saw in class, and you can see in the spreadsheet `BinomialTree.xlsx`, that the cash "position" in the dynamic replication of a call option can be negative. This means that you borrow money (to get a negative amount of cash) and use that to buy stock for the replicating portfolio.

   (a) Is this also true for put options? Is the cash position in the replicating portfolio negative? You may modify the spreadsheet to find the answer, but please explain your answer without reference to the spreadsheet.

   (b) (*Extra credit*) Does it ever happen that the cash position in the replicating portfolio for a call option is positive?

2. Consider a binomial tree model with n time steps going from $t = 0$ (the time "today") up to time T, which is the fixed expiration time. Use the formulas for $u = 1 + \sigma\sqrt{\Delta t}$ and $d = 1 - \sigma\sqrt{\Delta t}$ from Assignment 6. Let $S_{\max}$ and $S_{\min}$ be the largest and smallest prices that appear anywhere in the binomial tree. Show that these occur at time T. Note that if $T = t_n = n\Delta t$ and T is fixed, then $n \to \infty$ as $\Delta t \to 0$.

   (a) Show that $S_{\max} \to \infty$ and $S_{\min} \to 0$ as $\Delta t \to 0$ with T fixed.

   (b) Suppose the bond price at time t_k is found by $B_{k+1} = e^{r\Delta t} B_k$, starting with $B_0 = 1$. Is it true that $B_{\max} = B_n \to \infty$ as $\Delta t \to 0$? Note that $B_{k+1} > B_k$, so B_k increases at every time step. Also note that the number of increases, which is n, goes to infinity as $\Delta t \to 0$.

3. An option is "in the money" when it expires if its value then is not zero. We say that the option is *deep* in the money if the price of the underlier is far from the strike price, so the option is worth a lot. At time $t = 0$ we do not know for sure whether an option will expire in the money, but if K (the strike price) is far above or below S_0 (for a put or a call respectively) we know the option is very likely to expire in the money. These options are also called deep in the money. There is not much trading of deep in the money options for reasons we explore in this exercise. Suppose the option is a put with strike $K > S_{\max}$ or a call with strike $K < S_{\min}$ (the analyses are different but similar). Examine the numbers in the binomial tree for such an option. In particular:

(a) Determine the price of the option today as a function of r, Δt, σ, etc.

(b) Show that it is not necessary to re-balance during the lifetime of the option.

(c) Write a formula for the cash and stock positions of the replicating portfolio today ($t = 0$).

Now take the formulas for (a) and (c) and let $\Delta t \to 0$ with fixed K. We know, from Exercise (2), that the hypothesis $K > S_{\max}$ cannot be satisfied for fixed K, but ignore that fact. In the limit $\Delta t \to 0$, what is the limit of the hedge and the value of an "option" that you know in advance will be exercised?

(If two parties agree that one will buy an asset from the other at time T for a price K that is specified now ($t = 0$), that is a *forward contract*. If an option is so deep in the money that it is almost sure to be exercised, then it is essentially a forward contract. This exercise gives a pricing formula and the corresponding hedging strategy for a forward contract. You may have noticed that the volatility parameter σ didn't occur in your pricing formula. That's because the market price of the asset at time T is irrelevant.)

4. Suppose $p_u = p_d = \frac{1}{2}$ in the one step binomial model. There are up and down distances u and d so that

$$\left.\begin{array}{rcl} \mathrm{E}[S_1 - S_0] & = & \mu S_0 \Delta t \\ \mathrm{var}(S_1 - S_0) & = & \sigma^2 S_0^2 \Delta t \ . \end{array}\right\} \qquad (1)$$

For small Δt, these distances have the form

$$\left.\begin{array}{rcl} u & = & 1 + a_u\sqrt{\Delta t} + b_u \Delta t + \text{ order } \Delta t^{3/2} \\ d & = & 1 - a_d\sqrt{\Delta t} - b_d \Delta t + \text{ order } \Delta t^{3/2} \ . \end{array}\right\} \qquad (2)$$

Find formulas for a_u, a_d, b_u, and b_d in terms of μ and σ. The formulas do not involve Δt. Choose one of two possible ways to do this. One way is to find exact formulas for u and d that satisfy the conditions (1) and then write Taylor approximations to the desired order in the form (2). The other way is to substitute the approximate expressions (2) into the conditions (1) and solve for the coefficients a_u, a_d, b_u, and b_d directly.

Here's what you need to do it the second way. Suppose $x = 1 + c_x\sqrt{\Delta t} + d_x \Delta t + \cdots$ and $y = 1 + c_y\sqrt{\Delta t} + d_y \Delta t + \cdots$, then (calculating the square) $x^2 = 1 + 2c_x\sqrt{\Delta t} + (c_x^2 + 2d_x)\Delta t + \cdots$. A more complicated calculation is

$$\begin{aligned} e^{x-1} &= 1 + (x-1) + \frac{1}{2}(x-1)^2 + \cdots \\ &= c_x\sqrt{\Delta t} + \left(d_x + \frac{1}{2}c_2^2\right)\Delta t + \cdots \ . \end{aligned}$$

Also, $xy = 1 + (c_x + c_y)\sqrt{\Delta t} + (d_x + d_y + c_x c_y)\Delta t + \cdots$. If $x = y$ then $c_x = c_y$ and $d_x = d_y$. These are all Taylor approximations in the small parameter $\varepsilon = \sqrt{\Delta t}$. The rules are: (1) You calculate "up to" a given order by doing ordinary calculations but ignoring any power of Δt that is larger (e.g. $\Delta t^{3/2}$ is a larger power than $\Delta t = \Delta t^1$) than the order you are going up to. The calculations above are up to order Δt. (2) If two quantities are equal up to a given order, then the corresponding coefficients up to that order are equal.

For example, suppose we have equations

$$x + 2y = \Delta t$$
$$xy + \Delta t = 0 \ .$$

We assume a Taylor approximation in powers of $\sqrt{\Delta t}$ and substitute in. From the first equation we find

$$c_x\sqrt{\Delta t} + d_x\Delta t + 2c_y\sqrt{\Delta t} + 2d_y\Delta t = \Delta t \ .$$

We equate the coefficients of the powers of Δt, starting with the largest terms (which have the smallest powers). That's $\sqrt{\Delta t} = \Delta t^{1/2}$, and gives (because the coefficient of $\sqrt{\Delta t}$ on the right side is zero)

$$c_x + 2c_y = 0 \ .$$

This tells us that $c_x = -2c_y$. Then we equate the coefficients of the next power of Δt, which is $\Delta t^2 = \Delta t$. This gives

$$d_x + 2d_y = 1 \ .$$

At this point we haven't found the value of any of the coefficients, but we know relations between them. Now use the second equation. This gives

$$c_x c_y \Delta t + \cdots + \Delta t = 0 \ .$$

This tells us that $c_x c_y = -1$. We already know that $c_x = -2c_y$, so we now have $-2c_y^2 = -1$, or $c_y = \frac{1}{\sqrt{2}}$. Substitute back and you see that $c_x = \sqrt{2}$. At this point, we don't know d_x or d_y, but we do have a relation between them.

5. Write a script in `R` that uses the binomial tree algorithm to compute the price of a put or a call (you choose which). It should have parameters S_0 (spot price), T (time to expiration), r (the risk free interest rate), σ (the vol), and n (the number of time steps). The rest of the parameters, q_u, u, Δt, etc., should be calculated by the script in terms of these. Verify the program by checking that it gives the same result as the spreadsheet `BinomialTree.xlsx` with appropriate parameters. Show, computationally, that the limit $n \to \infty$ makes sense if the other parameters are fixed. Choose reasonable values of σ, r, S_0, K, and T. For example, $K = S_0$ is reasonable if T is not more than a year or two, r has some relation to present reality, σ is not in the hundreds, etc.