

Assignment 4, due October 6

Corrections: (Oct 1: Formula (1) fixed. Oct. 2: Question 1b corrected (wording change to clarify, $-d_2$ should have been d_2 in the formula for C), and correcting the reference in question 1e to week 4, page 2.)

1. (Black Scholes formula math)

- (a) Consider a portfolio that is long one call and short one put with the same strike, K , and expiration date, T . Show that at time T , the payout of the portfolio is $S_T - K$. Find the value today of that payout. Use this to get the *put call parity* formula

$$P(s, K, T, \sigma, r) = C(s, K, T, \sigma, r) - s + Ke^{-rT}. \quad (1)$$

- (b) Use the formula $N(d) = 1 - N(-d)$, together with put call parity (1) to get the formula (22) from the week 4 notes for P , starting with $C = sN(d_1) - Ke^{-rT}N(d_2)$.
- (c) Use (1) to show that $\partial_\sigma P = \partial_\sigma C$. Both are called *Vega* and written Λ . It is common to simplify partial derivative notation and write, for example, $\partial_\sigma P$ for $\frac{\partial P}{\partial \sigma}$.
- (d) Use the formulas

$$\begin{aligned} d_1 &= \frac{\log(M)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \\ d_2 &= \frac{\log(M)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} = d_1 - \sigma\sqrt{T} \end{aligned}$$

to get

$$N'(d_2) = MN'(d_1). \quad (2)$$

Here $N'(z) = \frac{dN}{dz} = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$. (I use the variable z to avoid writing $N'(d) = \frac{dN}{dd}$.)

- (e) Use $d'_1 = \partial_\sigma d_1$ and $d'_2 = \partial_\sigma d_2 = d'_1 - \sqrt{T}$ and (2) to get

$$\Lambda = s\sqrt{T}N'(d_1). \quad (3)$$

- (f) Find a formula for the largest possible Vega among all options with the same expiration date on the same underlier. It is curious that this pricing related formula involves $\sqrt{\pi}$.

(g) Suppose Π is any portfolio. The Vega of this portfolio is the derivative of its price with respect to σ . Let $\Pi(K)$ be a portfolio worth one unit of currency containing calls all struck at K and having the same expiration time. What values of K (near the money, deep in the money, deep out of the money) give the largest portfolio Vega? Hint: the formulas (3) and (4) from page 2 of the week 4 notes make this easy.

2. Figure out the limits

$$\lim_{\sigma \rightarrow 0} C(s, T, K, r, \sigma) , \quad \lim_{\sigma \rightarrow \infty} C(s, T, K, r, \sigma) .$$

Use the Black Scholes formulas (20) (21) (22) (23) on page 6 of the week 3 notes. For the $\sigma \rightarrow 0$ limit, it probably is easier to do the case in the money forward and out of the money forward (moneyness positive or negative) separately. The answers are simple. Give an interpretation of the $\sigma \rightarrow 0$ limit in terms of what the option would be worth if there were no uncertainty.

3. A European style *digital* option pays one unit if S_T is in a certain range and nothing otherwise. A simple one sided digital pays one unit if $S_T > K$ and nothing otherwise. Find a formula for $f(s, T, K, \sigma, r)$, the value of this options today if $S_0 = s$, etc. Calculate the integral (13) from page 4 of the week 3 notes.

4. We will compute and plot implied vol curves using Excel macros written in VBA. Download the Excel spreadsheet `a4.xls` from the homework web page. This has VBA macros `EPut`, which computes the Black Scholes put value, and `EPutIVol`, which computes the Black Scholes implied volatility for a European put. You will have to “enable macros” when you open `a4.xls` in Excel, but I promise that I did not put a virus in it. To see the VBA code, in Excel, click on “Tools”, then “Macro”, then “Visual Basic Editor”, then in the “Projects” window open the “Modules” folder (if it isn’t already open) and doubleclick on “Module 1”. If everything is working correctly – *please post ASAP on the class message board otherwise* – the price in cell `C9` should be (almost) the same as the price in cell `B6`. The sheet is programmed to compute the implied vol for given put parameters and put the result in cell `C8`. It then computes the price for the vol in cell `C8`, which goes into cell `C9`. Try changing the put price `B6` to see that the implied vol calculator is working correctly.

(a) Create a new macro called `ECall` that computes the Black Scholes value for a European call. It should be easy to copy and modify the code for `EPut`. You only have to change the next to last line. The clunky statement `WorksheetFunction.NormSDist(d)` in VBA produces the same result as `NormSDist(d)` in Excel, and the same for many other Excel built in functions (but not `Exp`, for example). Call your `ECall` from `a4.xls` and check that the result is correct.

- (b) Copy and modify the VBA code for `EPutIVol` to make a macro `ECallIVol` that computes implied vol for a European Call. Test that it produces the right answer in the way the put implied vol code is tested in the posted `a4.xls`.
- (c) You have to fix one oversight in `a4.xls` as posted. The formula in cell `C8` refers to its parameters by cell references (`B1`, `B2`, etc.). This means that when you copy and paste the formula in this cell, it will look for its arguments in the wrong places. Try copying cell `C8` and pasting into cell `E8`, you will get an error. At least I do. You can fix this by assigning names to values in parameter cells then referring to the names rather than the cell addresses. For example, you can tell Excel that the variable `S0` has value given by cell `B1` as follows: Click on cell `B1` and somewhere in the top left the *name box* says `B1`. Click in this name box and type `S0`. That assigns the name `S0` to cell `B1`. You can do the same for `T` and `r` and `K`. Now you can do as in assignment 1: create a column of strikes and market put prices (say, strikes in column `D` and prices in column `E`, then in cell `F2` compute implied vol from `D2` and `E2` (putting column labels in `D1` and `E1`), then copy down to do the rest of them automatically.
- (d) Download the files `FXOptionData.xls` and `EquityOptionData.xls` and play with the data. There are three categories: an equity index (the S&P500), an individual large cap name (Exxon), and FX (USD/Euro exchange rates). For the FX, there are 6 month and shorter dated options because there is little volume on the six month options. In general (according to Hull and the data here), most FX option trading is OTC.

Look at the shapes of the implied vol curves from these prices. Use mostly out of the money and not far in the money options. Deep in the money options hardly trade because they are not very interesting (being almost the same as forward contracts and lacking convexity). Many of the prices listed are not “real” in the sense that there is no actual trading at those prices. Real numbers are characterized by significant volume and narrow bid/ask spreads. You will have to judge which prices to use. It is common to take the midpoint between the bid and ask prices when computing implied vol, though this should not make much difference.

Prepare a few graphs of computed implied vols. Explain your choice of data. Describe the skew and smile of the graphs and the differences between them. This is in part a writing exercise. You need to explain your methodology clearly. I particularly encourage collaboration on this, as you will benefit from hearing other people’s thinking and reasoning. This item is much less well defined than the others. Please think about, present, and defend your choices.

- 5. Use the binomial tree pricer `bintree.cpp` from last week to verify that the binomial tree price of a near the money call converges to the Black Scholes

price as $\delta t \rightarrow 0$ with the binomial tree parameters chosen appropriately. Present a sequence of computed values with $T = 1$ (one year, typically) and $n = 5, 10, 20, 40,$ and 80 steps. Compare the binomial tree values with the exact Black Scholes value. Take the parameters in the `a4.xls` spreadsheet, except replace the silly 5% risk free rate with a more realistic number. You can get the parameters for the binomial tree from the week 4 notes or an exercise from assignment 3. Explain how you did this.