

Assignment 9, due November 19

Corrections: (none yet)

1. Read *Conversion Factors* in Section 6.2 of Hull. Suppose a futures contract calls for delivery of a bond with at least 20 years remaining to maturity. Assuming the bond expires exactly 20 years after the futures settlement date, what coupon would it have to pay for its duration to be less than 18 years?
2. Consider a zero coupon bond that pays one unit of currency at time T_1 and a European option to buy this bond at time $T < T_1$ for price K . Using Black's model, (Kohn's notes, Section 10, pp. 1-3) is there an exercise time, T that maximizes the price of the option at time $t_0 = 0$? What is the behavior of this price as a function of T (the exercise time) as $T \rightarrow 0$ and as $T \rightarrow T_1$? Give the limiting value and state whether it is increasing or decreasing? How do your answers depend on K ?
3. Consider a risk neutral world where the money market $M(t)$ is the numeraire ($dM(t) = r(t)M(t)dt$, $M(0) = 1$). Here, $r(t)$ is the short rate at time t . Let $B(t, T)$ be the zero coupon bond price for borrowing at time t and repaying at time T .

(a) Show that

$$B(t, T) = E_{RN} \left[\frac{M(t)}{M(T)} \middle| \mathcal{F}_t \right].$$

Here, $E_{RN}[\cdot]$ means expectation in the risk neutral world. The notation \mathcal{F}_t means all information about the world between times 0 and t . In the present case, this is the same as the expected value given that the values $M(t)$ and $r(t)$ are known:

$$B(t, T) = E_{RN} \left[\frac{M(t)}{M(T)} \middle| M(t), r(t) \right].$$

In both cases, the values $M(t+s)$ and $r(t+s)$ for $s > 0$ are unknown.

(b) Show from this that $B(t, T) = B_*(r(t), t, T)$, where

$$B_*(r, t, T) = E_{RN} \left[\exp \left(- \int_t^T r(s) ds \right) \middle| r(t) = r \right].$$

- (c) Suppose that in this world, $r(t)$ satisfies an SDE of the form

$$dr = a(r_*(t) - r(t))dt + \sigma dW(t),$$

where $r_*(t)$ is either a constant (as in the Vasicek model) or a known and not random function of t (Hull and White, one factor model). Show that $r(t)$ and $\int_t^T r(s)ds$ are Gaussian random variables.

- (d) Conclude that in the risk neutral world, if the short rate process is Vasicek or Hull White, then $B(t, T)$ is indeed a lognormal random variable.
- (e) Find the mean and variance of $\log(B(t, T))$ in the case $dr = \sigma dW$. From this, it should be clear that you could calculate the mean and variance for the Hull and White one factor model, though the computations would be lengthy.
- (f) Compare the form of the variance of $\log(B(t, T))$ as a function of t to the form suggested in Kohn's notes. Are the qualitative behaviors similar as $t \rightarrow 0$ and $t \rightarrow T$?
- (g) Does that imply that Black's model gives the correct prices for bond options if the risk neutral world has a simple short rate process as in this problem?
4. Use Excel to create a spreadsheet that computes the present price of a $T =$ ten year interest rate cap with cap rate R_K using Black's formula. The total price is the sum of twenty caplet prices. Make a plot of the price of the cap as a function of R_K . Determine the interesting range of R_K values and explain what happens for small R_K and large R_K . Suppose that the principal is one unit of currency and that the payments are semi-annual at the end of each half year using the rate at the beginning of the half year (if not capped). Assume a volatility $\sigma^2 T$ with $\sigma = .2$. Use the following prices for zero coupon bonds (presumably found from market coupon bonds via bootstrapping): You will need to construct bond prices

maturity (years)	.5	1	2	3	5	10
discount	.9792	.9567	.9057	.8537	.7461	.5612

$B(0, T)$ for times T not given in in the above table. For that purpose, construct first a yield curve, then get the yield for intermediate times by linear interpolation. Hand in a printout of the spreadsheet as well as the plot.

It will be important in this problem to plan the spreadsheet construction to minimize the amount of work you do by hand. Hand in a description of how you constructed the spreadsheet.