Honors Algebra II, Courant Institute, Spring 2020

http://www.math.nyu.edu/faculty/goodman/teaching/HonorsAlgebraII2020/HonorsAlgebraII.html

Always check the classes message board before doing any work on the assignment.

Assignment 4, due February 24

Corrections: [none yet]

There is a short quiz in class on February 24. The point of the quiz is to see that you are following the material, and partly to see where you are in the kind of reasoning involved in abstract algebra. It is more of a reality check than a competitive exam. Here are some questions to give you some idea what to expect. The quiz itself will be short and have less questions than this. These questions are meant as a study aid. Do not hand in solutions.

1. Give an example of a rational prime (an integer that is a prime number) that is not prime in $\mathbb{Z}[i]$. Explain your answer.

2. Give an example of a field extension $\mathbb{E}/\mathbb{Q}$ that is not a Galois extension. Explain your answer. (What is a Galois extension? How do you know this isn't one?)

3. Suppose $A\colon V \to V$ is a linear map on the vector space V over a field $\mathbb{F}$. Suppose $\ker(A)$ is trivial.

   (a) Give an example that shows A need not be onto.

   (b) Give a hypothesis on V (not A) that implies that A is onto.

4. Suppose V and W are finite dimensional vector spaces over field $\mathbb{F}$. Suppose $A\colon V \to W$ and $B\colon W \to V$ are linear maps with $AB = \mathrm{id}_W$.

   (a) Give an example with $BA \neq \mathrm{id}_V$.

   (b) Give a hypothesis on $\dim(V)$ and $\dim(W)$ that implies that $BA = \mathrm{id}_V$.

5. Give an example of an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree 4 so that the splitting field of f over $\mathbb{Q}$ is a degree 4 extension.

6. Suppose $\mathbb{E}/\mathbb{F}$ is a finite degree field extension. Let f be an irreducible polynomial in $\mathbb{F}[x]$ and have a root $\alpha \in \mathbb{E}$. Let $b_1, \ldots, b_n$ be a basis of $\mathbb{E}$ over $\mathbb{F}$.

   (a) Show that the map $x \to \alpha x$ is a linear map of $\mathbb{E}$ as a vector space over $\mathbb{F}$.

   (b) Let M_α be the matrix of this linear map in the basis b_k. Show that α is an eigenvalue of of M_b and describe a corresponding eigenvector.

7. Is there a basis of $\mathbb{Q}[2^{\frac{1}{3}}]$ over $\mathbb{Q}$ that has no rational basis elements? Either exhibit such a basis or show that it cannot exist.

These are the usual exercises. Please hand in solutions to these as usual.

1. Let $\mathbb{E}$ be a normal extension field of $\mathbb{F}$. Let $f \in \mathbb{F}[x]$ be irreducible in $\mathbb{F}$. Show that if f has a root in $\mathbb{E}$, then f splits in $\mathbb{E}$. Hint, let $\xi_1, \ldots, \xi_m$ be all elements of $\mathbb{E}$ of the form $\xi_j = \sigma(\xi)$ with $\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. *Hint*: Show that $g(x) = (x - \xi_1) \cdots (x - \xi_m)$ is a polynomial with coefficients in $\mathbb{F}$ that divides f.

2. Let $\mathbb{E}/\mathbb{B}$ be a normal extension of degree 2^n. Show that there is a tower of fields $\mathbb{B} = \mathbb{E}_0 \subset \mathbb{E}_1 \subset \cdots \mathbb{E}_n = \mathbb{E}$ where $\mathbb{E}_{k+1}/\mathbb{E}_k$ is degree 2 and normal.

3. Show that if $\mathbb{B}/\mathbb{F}$ is an algebraic field extension, not necessarily normal, then there is an irreducible polynomial $f \in \mathbb{F}[x]$ that has a root in $\mathbb{B}$ but not in $\mathbb{F}$.

4. Show that if $\mathbb{E}/\mathbb{F}$ is a normal extension of finite index m not a power of 2, then there is an irreducible polynomial $f \in \mathbb{F}[x]$ of odd degree greater than one that splits in $\mathbb{E}$. *Hint*: If m is even, a Silow theorem gives you a subgroup, $H \subseteq \mathrm{Gal}(\mathbb{E}/\mathbb{F})$, not necessarily normal, with $|H| = 2^k$.

5. Show that if $\mathbb{E}/\mathbb{Q}$ (with $\mathbb{E} \subset \mathbb{C}$) is a normal extension of degree not a power of 2, then $\mathbb{Q} \neq \mathbb{E} \cap \mathbb{R}$. You may use the intermediate value theorem applied to a polynomial of odd degree.

6. Show that if $\mathbb{E}/\mathbb{R}$ is a finite degree normal proper extension, then $\mathbb{E}$ is isomorphic in the category of fields to $\mathbb{C}$. This is a Galois theory proof of the "fundamental theorem of algebra" (which actually is a theorem of analysis – the intermediate value theorem). This argument using exercises 2, 3, and 4 was suggested to me by Professor Spencer.

Algebraic number theory has many deep applications of Galois theory. Assignments will often feature facts from ring theory relevant to algebraic number theory. The Gaussian integer exercises were part of this.

7. Let R be a ring without *zero divisors* ($x \neq 0$ is a zero divisor if there is a $y \neq 0$ with $xy = 0$. This could, be written $y = 0/x$ (but should not be), which would make x a divisor of 0.). An ideal $\mathfrak{p} \subset R$ is *prime* if $xy \in \mathfrak{p}$ implies that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

   (a) Show that if R is a unique factorization domain (every $x \in R$ factorizes uniquely into irreducibles modulo units), and if $f \in R$ is irreducible, then the ideal (f) is a prime ideal.

   (b) We saw in assignment 3 that the ring $R = \mathbb{Z}[i\sqrt{5}]$ does not have unique factorization into irreducibles. Show that the ideal (2) is not a prime ideal in $\mathbb{Z}[i\sqrt{5}]$.

(c) How general is this phenomenon (principal ideals on irreducibles not prime ideals) in the category of integral domains that do not have unique factorization into irreducibles?