Assignment 11, due April 20

Corrections: April 13: Exercise 7 replaced with something about representations mod p. April 22 (due date, I'm sorry), Exercise 7 fixed and simplified. The original version was wrong.

The first series of exercises is on Jordan form of a matrix or linear transformation. A *Jordan block* with eigenvalue λ and size k is a $k \times k$ matrix with λ on the diagonal and 1 on the *superdiagonal*. Matrix entries just above the main diagonal are in the *superdiagonal*. An element a_{kk} is on the diagonal, and $a_{k,k+1}$ is in the superdiagonal.

$$B_{\lambda,k} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & 1 \\ 0 & \cdots & & 0 & \lambda \end{pmatrix} \;, \quad (k \times k \text{ matrix}).$$

We use $J_{k\times k}$ to denote the matrix with ones on the super-diagonal:

$$J_{k\times k} = B_{0,k} \;.$$

This matrix acts on a column vector by shifting the components up:

$$J_{k\times k}\, x = J_{k\times k} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ x_k \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_k \\ 0 \end{pmatrix} \;.$$

The component x_1 is lost. A zero is "shifted" in as the last component. A Jordan block can be written in terms of the $k \times k$ identity matrix as

$$B_{\lambda,k} = \lambda I_{k\times k} + J_{k\times k} \;. \tag{1}$$

An $n\times n$ matrix A is in *Jordan form* (or Jordan *normal* form, or Jordan *canonical* form) if it is block diagonal with Jordan blocks on the diagonal. That means that there are "numbers" (elements of a field K) λ_j and sizes k_j with $n = k_1 + \cdots + k_m$.

$$A = \begin{pmatrix} B_{\lambda_1,k_1} & 0 & \cdots & 0 \\ 0 & B_{\lambda_2,k_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & B_{\lambda_m,k_m} \end{pmatrix} \;. \tag{2}$$

The number of blocks is m.

Let V be vector space of dimension n over an algebraically closed field K, with $\rho\colon V \to V$ a linear transformation. The Jordan form theorem is that there is a basis of V in which ρ is represented by a matrix in Jordan form. The basis need not be unique and the blocks may be in any order. The eigenvalues λ_j and the block sizes k_j are uniquely determined by ρ.

The linear transformation is *diagonalizable*, and the Jordan form (2) is diagonal, if all the blocks have size $k = 1$. Otherwise, A has *non-trivial Jordan structure*. Let $p(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of A. If p has n distinct roots $\lambda_1, \ldots, \lambda_n$, then the block sizes are all $k_j = 1$, because n positive integers k_j cannot add up to n unless each is equal to 1. If $p(\lambda_1) = p'(\lambda_1) = 0$ (λ_1 is not a simple root), then it is likely, but not necessary, that there is non-trivial Jordan structure, $k_1 > 1$, corresponding to that eigenvalue.

1. A linear transformation $\rho : V \to V$ is *nilpotent* if there is an m with $\rho^m = 0$. A nilpotent matrix that is diagonalizable must be the zero matrix. Suppose $\rho \neq 0$ but $\rho^2 = 0$. This exercise shows that ρ has non-trivial Jordan structure and explains how to find a basis in which ρ has Jordan form. The point is to find basis vectors x and y with $\rho x = 0$ and $\rho y = x$. The vector y has $\rho y \neq 0$ but $\rho^2 y = 0$. The vector x is an eigenvector of ρ with eigenvalue $\lambda = 0$. The vector y is a *generalized* eigenvector, also associated with eigenvalue $\lambda = 0$. For higher powers $m > 2$, some *Jordan chains* will be longer, taking the form

$$x_m \xrightarrow{\rho} \rho x_m = x_{m-1} \xrightarrow{\rho} \cdots \xrightarrow{\rho} \rho x_2 = x_1 \neq 0 \xrightarrow{\rho} \rho x_1 = 0 \;.$$

A chain like this is consistent with $\rho^m = 0$ but $\rho^{m-1} \neq 0$. The Jordan form theorem for nilpotent matrices says that V has a basis consisting of chains like this. Different chains can have different lengths, but the longest chain has a length equal to the highest power m.

For example, consider the nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors $e_3 \to e_2 \to e_1 \to 0$ form a chain of length 3, and the chain $e_5 \to e_4 \to 0$ has length 2. This matrix is in Jordan form (2) with a 3×3 block and a 2×2 block, both with eigenvalue 0. It has $A^3 = 0$ but $A^2 \neq 0$.

Throughout this exercise, take $m(\rho)$ to be the smallest integer with $\rho^m = 0$. This has $\rho^{m-1} \neq 0$. Take $n = \dim(V) < \infty$. This exercise finds the Jordan structure of a nilpotent transformation, which is the technical core of the Jordan form theorem. The strategy is to find the "end" vectors, which are true eigenvectors, then to find generalized eigenvectors that map to these true eigenvectors. Some eigenvectors will be "hit" in this way and others may not be, depending on the Jordan structure. Then we find whatever vectors may map to the first generalized eigenvectors, and so on. The main technical idea is part 1d.

(a) Show that a $\lambda = 0$ Jordan block $B_{0,k}$ has $m = k$.

(b) Define $V_j = \ker(\rho^j)$ with dimension d_j. Show that $V_{j-1} \subset V_j$ and that each containment is strict in the sense that $1 \le d_1 < d_2 \cdots < d_m = n$. *Hint*: Show that ρ takes V_j onto $V_{j-1} \subseteq V_j$. Let ρ_j be the restriction of ρ to V_j. Show that if $d_{j-1} = d_j$ then $V_{j-1} = V_j$ and ρ_j is an automorphism and not nilpotent.

(c) Show that if $m = 1$ then ρ is diagonalizable. (This is trivial but good to remember.)

(d) Suppose $m = 2$. Construct a basis of V consisting of three parts. The x_j, for $j = 1, \ldots, r$, are a basis for $W_1 = \rho(V_2) \subset V_1$. The y_j, for $j = 1, \ldots, s$, extend the x_j so that the x_j and y_j together are a basis for V_1. The z_j, for $j = 1, \ldots, r$ are chosen arbitrarily so that $\rho(z_j) = x_j$. Show that these vectors together form a basis for V. *Hint*: To see they are linearly independent, consider a linear combination in V of the form

$$0 = \sum_{j=1}^{r} a_j x_j + \sum_{j=1}^{s} b_j y_j + \sum_{j=1}^{r} c_j z_j \ .$$

If you apply ρ and use the fact that the x_j are linearly independent, you see that the c_j must be zero. For any $u \in V_2$, you can write $\rho u = \sum c_j x_j$ (why?) and see that $u - \sum c_j z_j \in V_1$ (why)? [There was an argument like this in our work on noetherian modules. If $N \subset M$ is noetherian and M/N is noetherian, then M is noetherian. We took a finite basis of M/N and chose arbitrary elements in M that map to them.]

(e) Generalize the argument of the above two parts to $m = 3$ or higher. You may do just the case $m = 3$, since larger m is the same, but with more notation. Show that a 3×3 Jordan block arises from a triple of basis elements $u \xrightarrow{\rho} v \xrightarrow{\rho} w \xrightarrow{\rho} 0$.

(f) Show that for $m = 2$, the basis constructed in part (1d) represents ρ with a matrix in Jordan form with r 2×2 blocks and s 1×1 blocks.

(g) The characteristic polynomial is $p(\lambda) = \det(\lambda I - \rho)$. Show that $p(\lambda) = \lambda^n$. This shows that different Jordan structures are compatible with the same characteristic polynomial. *Hint*: A fancy argument uses the fact that p splits in the algebraic closure of K an that a nilpotent transformation cannot have $\rho x = \lambda x$ with $\lambda \neq 0$. This exercise gives a more elementary yet more complicated proof.

2. Let ρ act on V but do not assume ρ is nilpotent. The *nil subspace* of V is W_1, which is the set of $x \in V$ with $\rho^j x = 0$ for some $j > 0$. We want to write V as a direct sum of subspaces corresponding to different eigenvalues. The nil subspace W_1 (or generalized eigenvalue subspace) corresponds to eigenvalue $\lambda = 0$. The point is that there is a complementary space W_2 that is also invariant under ρ.

   (a) Show that there is an m with $W_1 = \ker(\rho^m)$. Show that W_1 is an invariant subspace (stable subspace) under ρ. Define $\rho_1 : W_1 \to W_1$ to be the restriction of ρ to W_1.

   (b) Define $W_2 = \rho^m(V)$. Let ρ_2 be the restriction of ρ to W_2. Show that ρ_2 is invertible on W_2 and that W_2 is an invariant subspace under ρ. *Hint*: Tricks from part (1d) may apply.

   (c) Let $x_1, \ldots, x_r$ be a basis for W_1 and $y_1, \ldots, y_s$ be a basis of W_2. Show that $x_1, \ldots, x_r$ be a basis for W_1 and $y_1, \ldots, y_s$ form a basis for V and that $V = W_1 \oplus W_2$ with $\rho = \rho_1 \oplus \rho_2$.

   (d) Show that ρ_1 on W_1 has a Jordan structure.

3. Show that if λ is a root of the characteristic polynomial, then $\rho - \lambda I$ has a non-trivial nil subspace. Call this subspace V_λ. Show that V_λ is stable under ρ, that V is a direct sum of such subspaces. Show that you can combine bases of these V_λ to find a basis for V in which ρ has the form (2).

4. A chain of *generalized eigenvectors* for eigenvalue λ is a sequence of non-zero vectors x_j so that

$$x_j \overset{A-\lambda I}{\rightarrow} x_{j-1} \overset{A-\lambda I}{\rightarrow} \cdots \overset{A-\lambda I}{\rightarrow} x_1 \overset{A-\lambda I}{\rightarrow} 0 \,.$$

   Of course, the last vector, x_1 is a true eigenvector. Show that the basis in which ρ has the Jordan normal form is a basis consisting of generalized and true eigenvectors.

5. Consider the $n \times n$ matrix

$$A = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & & -a_0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & 0 & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} .$$

   Consider the polynomial $p(x) = x^n + \cdots + a_0$. The matrix A is the *companion matrix* for p.

(a) Show that if $p(\lambda) = 0$, and

$$x = \begin{pmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ 1 \end{pmatrix} ,$$

Then x is an eigenvector of A with eigenvalue λ.

(b) Show that if λ is a double root of p ($p(\lambda) = 0$, $p'(\lambda) = 0$) then

$$y = \frac{d}{d\lambda} x = \begin{pmatrix} (n-1)\lambda^{n-2} \\ \vdots \\ 0 \end{pmatrix} ,$$

generalized eigenvector with $y \overset{A-\lambda I}{\to} cx \overset{A-\lambda I}{\to} 0$.

6. Suppose G is an infinite group and $\rho\colon G \to \text{aut}(V)$ is a finite dimensional representation. Here, $\text{aut}(V)$ is the group of linear automorphisms (invertible linear maps) of V. The representation is *simple* if there is no proper invariant subspace $W \subset V$ so that $\rho(g)\colon W \to W$ for all $g \in G$. The representation is *semi-simple* if it is a direct sum of simple representations. Theorem 2 of *Linear Representations of Finite Groups* states that any representation of a finite group on a vector space over $\mathbb{C}$ is semi-simple. Take $G = \mathbb{Z}$, let A be an $n \times n$ non-singular matrix.

   (a) Show that $n \to A^n$ defines a linear representation of $\mathbb{Z}$ on $V = \mathbb{C}^n$. The representation may be denoted (with a slight abuse of notation) by $\rho_A(n) = A^n$.

   (b) Show that ρ_A is semi-simple if and only if A is disgonalizable over $\mathbb{C}$ (non-trivial Jordan block $\Longrightarrow$ not semi-simple).

   (c) Give an example of a real matrix A that defines a real (not complex) representation ρ_A over $\mathbb{R}^n$ that is semi-simple but A is not diagonalizable (over $\mathbb{R}$).

   (d) (*extra credit*) Suppose A and B are invertible commuting $n \times n$ matrices. Show that $\rho_{AB} : (m, n) \to B^m A^n$ defines a representation of $\mathbb{Z}^2$ on $\mathbb{C}^n$.

   (e) (*extra credit*) Show that this representation is semi-simple if and only if A and B are *simultaneously diagonalizable* over $\mathbb{C}$. This means that there is a single invertible $n \times n$ complex matrix P so that $P^{-1}AB$ and $P^{-1}BP$ are both diagonal.

7. Consider the 2 dimensional representation ρ with representation space $V = \mathbb{F}_p^2$ over the finite field $\mathbb{F}_p$. The group is the cyclic group $G = C_p = \{0, 1, \ldots, p-1\}$. For 1 the generator of C_p, the representation $\rho(1)$ acts on (x, y) by $y \to y$ and $x \to x + y$. This exercise shows that this finite dimensional representation of a finite group is not semi-simple. It's basically a Jordan block for eigenvalue $\lambda = 1$.

(a) Show that for any pair of communing matrices the binomial formula holds and identify the precise upper and lower limits for the summation:

$$(B+C)^k = \sum_j \binom{k}{j} B^j C^{k-j} \ . \tag{3}$$

(b) Identify the matrix $A(1)$ that represents $\rho(1)$ in the basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $(x, y) = xe_1 + ye_2$. Write $A(1)$ in the form $I + J$ of (1). Use the binomial formula (3) to calculate $A(k) \in \text{aut}\big(\mathbb{F}_p^2\big)$. Show that $k \in C_p$ the map $k \to A^k$ is well defined and is a representation of C_p. *Hint*: The calculation is related to the calculation that shows $(x+y)^p = x^p + y^p$ in $\mathbb{F}_p$.

(c) Verify (along with the precise start and end values in the sum) the formula

$$A^k = \sum_j \binom{k}{j} \lambda^j J^{k-j} \ .$$

(d) Show that the subspace $W \subset V$ generated by e_2 is invariant under ρ.

(e) Show that there is no other one dimensional subspace W' that is invariant under ρ. *Hint*: If W' is generated by $u \in \mathbb{F}_p^2$, then $\rho(1)u = cu$ for some $c \in \mathbb{F}_p$ (why?). Check whether this is possible with $u = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x \neq 0$.

(f) (*extra credit*) Let $V_R = \mathbb{F}_p^p$ be the representation space for the regular representation of C_p over $\mathbb{F}_p$. Show that V_R is semi-simple. *Hint*: Look for p one dimensional representations and show they all are direct summands in V_R.

(g) (*more extra credit*) We saw that over $\mathbb{C}$ every basic representation has an isomorphic copy inside the regular representation. This exercise shows that is not always true over other fields. Show that the two dimensional representation above does not have an isomorphic copy inside of V_R. *Hint*: You know where e_2 would have to go, but what about e_1?