Balancing Small Transaction Costs with Loss of Optimal Allocation in Single and Multiple Stock Portfolios

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Abstract

We discuss optimal trading strategies in the presence of small proportional transaction costs for general utility functions. We present a new interpretation of scalings found by Soner, Shreve, and others. To leading order in the small transaction cost parameter, the free boundary problem for the expected utility's value function is shown to be dual, in the sense of Lagrange multipliers for optimal design problems, to a free boundary problem minimizing a cost function. This cost function is the sum of a boundary integral corresponding to the rate of trading and an interior integral corresponding to opportunity loss that results from suboptimal portfolio allocation. Using the dual problem's formulation, we show that the quasi-steady state probability density of the optimal portfolio is uniform for a single stock, but generally blows up even in the simple case of two uncorrelated stocks.

1 Introduction

Trading strategies that take into account small market imperfections, such as transaction costs, market discreteness, and limited market liquidity, can be expressed as perturbations of an idealized trading strategy derived without these market imperfections. The idealized, or *outer*, strategy could come from a Merton optimal problem, Black-Scholes delta hedging, etc. The *inner* trading problem is to find strategies for markets with small frictions that approximate the idealized strategy. The inner strategy must balance the loss created by market frictions with the opportunity loss created by the portfolio deviating from its idealized strategy.

The inner strategy is derived under simplifying assumptions that are justified if market frictions are small. Small market frictions imply the *imbalance*, that is, the difference between the actual position and the optimal position under the outer strategy, should also be small. To leading order, this yields many important idealizations for the inner problem. Many details of the inner problem now become irrelevant such as, in our case, the nature of the drift term since it becomes dominated by the diffusion due to Brownian motion. Further, since the imbalance stays within a small domain, it equilibrates quickly. Therefore, as the outer problem's market conditions change slowly, the inner imbalance at any particular time is approximately in steady state for those outer conditions. This will allow us to see the optimal inner strategy as an equilibrium between the steady state rate of transaction costs with the steady state rate of opportunity loss.

In this paper our outer trading strategy is the Merton strategy for optimally rebalancing a portfolio of stock(s) and cash under the idealized market of Merton, Black, and Scholes [1]. To this ideal market, we add proportional transaction costs (see, for example, Magill and Constantinides [2]) where we lose a fraction proportional to a small parameter, ϵ, of the value of each trade we make. For a portfolio with a single stock, the problem of proportional transaction costs has been studied by Shreve, Soner, and Janecek [3], [4], who applied viscosity solution methods (see Davis, et. al. [5]) to the case of power law and logarithmic utility functions, and also by Whalley and Wilmott [6], who employed asymptotic analysis to the case of exponential utility functions. Both groups found that transaction costs led to a minimal loss in expected utility on the order of $\varepsilon^{2/3}$ and that the ideal trading strategy allowed the portfolio to move freely within the interior of a *hold region*, $\mathcal{H}$, that stretches a distance γ, which is proportional to $\varepsilon^{1/3}$, about its center, the idealized Merton strategy. (Note that in other papers, the *hold region* is sometimes called the *no transaction region*.)

Our work tries to explain these earlier results by studying the trading strategies they imply. For a general utility function, we express the leading order loss in the expected utility due to the presence of transaction costs as an explicit tradeoff between two terms: (1) the *opportunity loss* due to the portfolio being allowed to stray from its optimal Merton location, which is a loss of order γ^2, and (2) the *trading loss* due to trades on the boundary of the hold region, which is a loss of order ε/γ. Optimizing $\gamma^2+\epsilon/\gamma$ with respect to γ yields the exponents 2/3 and 1/3 above. Keeping track of the coefficients in front of these two loss terms, we will get full expressions for both the loss in expected utility and the length γ for any utility function. We also give a duality argument that shows our method is equivalent to (the generalized form of) the asymptotic analysis used in [6].

Further, we show the duality argument and the expressions for the opportunity loss and the trading loss are extendable to the case of portfolios with multiple stocks. These expressions depend upon u, the quasi-steady state probability density function in the hold region. In the single stock case, u is uniform over the hold region. This is almost never true for multiple stocks. Paradoxically, as we will show, u becomes infinite at some points of the boundary of the hold region even though the trading cost loss rate is proportional to an integral of u over this boundary. We have been unable to determine the precise shape of the hold region or the minimum loss in expected utility in the general multiple stock case. For uncorrelated stocks, Atkinson and Mokkhavesa [7] showed the hold region is a rectangular box, but this is not the case for correlated stocks. Numerical computations by Muthuraman and Kumar [8] and Atkinson and Ingpochai [9] suggest that for correlated stocks, the hold region is a smoothly distorted rectangular box.

We note that in Shreve et. al. and Atkinson et. al., consumption occurs continuously over an infinite time horizon while in Whalley et. al. and our paper, consumption occurs only at the end of a finite time horizon. Either consumption model yields similar results. (Compare, for example, our single stock and uncorrelated multiple stock results with the more recent results obtained for the continuous consumption model via asymptotics and viscosity solution methods by Atkinson et. al. in [10] and [7].) Note also that any initial transaction to adjust the portfolio to an optimal balance of stock and cash and, in the finite time horizon model, any final transaction to liquidate the portfolio into cash create a loss of order ε and therefore are irrelevant to the analysis since this loss is of higher order than the order $\varepsilon^{2/3}$ loss of utility that occurs over time.

Let us outline our results a little more precisely, beginning with the single stock case. The outer problem, described in Section 2.1, is one of Merton's optimal dynamic investment problems. Specifically, we assume that our stock evolves by geometric Brownian motion with constant expected return, μ, and volatility, σ, while our cash postion has a constant return rate, r. At time t, the portfolio contains shares of stock worth a value of $X(t)$ and cash worth a value of $Y(t)$. The total value of the portfolio is $Z(t) = X(t) + Y(t)$. The goal is to choose an adapted investment strategy, $X(t)$, that maximizes the expected utility at a final time, T:

$$f^0(z,t) = \sup E_{z,t}\left[U(Z(T))\right] \ . \tag{1}$$

Here $U(z)$ is an increasing, concave utility function and $Z(t) = z$ where $t \leq T$. The Hamilton-Jacobi-Bellman equation for f^0 yields the optimal strategy: $X(t) = m(Z(t),t)$ for the determined *Merton* function, $m(z,t)$.

The inner problem minimizes loss rates (here, loss of expected utility due to the presence of transaction costs) in a statistical steady state. Let $\xi(t) = X(t) - m(Z(t),t)$ be the *imbalance variable.* If the hold region is small, we expect ξ at any given time to be roughly in a probabilistic steady state, given that diffusion processes reach equilibrium quickly in small domains. The inner problem is to adjust the boundary of the hold region so as to minimize the steady state rate of loss of utility due to the combination of trading costs and opportunity loss described above. This hold region shape depends on some parameters determined by the solution to the outer optimization problem.

In our proportional transaction cost model, we lose the fraction εb of the value of all stock purchases and the fraction εc of the value of all stock sales, where ε is a scaling factor. Of course, following the Merton strategy exactly leads to infinite transaction cost in this model. Adopting the notation in, for example [5], we let $L(t)$ be the total value of stock purchases up to time t and $M(t)$ be the total value of stock sold, which are nondecreasing functions of t. A purchase of an additional dL of stock causes $Y \to Y - dL$ and $X \to X + (1-\varepsilon b)dL$, so that $Z \to Z - \varepsilon b dL$. Similarly, the sale of dM of stock causes $Z \to Z - \varepsilon c dM$.

We find (see Section 2.2) that the imbalance satisfies

$$d\xi \approx dL - dM + a(z,t)dB \ , \tag{2}$$

where B is Brownian motion, and the diffusion related coefficient, a, depends only on σ, m, and $m_z = \frac{\partial}{\partial z}m(z,t)$. The hold region, $\mathcal{H}$, corresponds to $\xi \in [\beta,\gamma]$. Assuming $\mathcal{H}$ is small, we view a as being constant within it. Also, ξ will be approximately in steady state, and we let $u(\xi)$ be the steady state probability density. Accepting the common view that the optimal strategy is to buy or sell only at the boundary of the hold region (this was proven to be the case in the problem studied by Soner and Shreve), we suppose that $dL \neq 0$ only when $\xi = \beta$ and $dM \neq 0$ only when $\xi = \gamma$. This implies that

$$\frac{1}{2}a^2 u_{\xi\xi} = 0 \ , \quad \text{for } \beta < \xi < \gamma \tag{3}$$

and

$$u_\xi = 0 \ , \quad \text{when } \xi = \beta \text{ or } \xi = \gamma. \tag{4}$$

These conditions describe the behavior of u as a function of ξ for each fixed value of the outer variables z and t. The steady state rate of loss in expected utility is, to leading order,

$$E\left[df^0(Z(t),t)\right] \approx -\varepsilon f_z^0 \left(bE[dL] + cE[dM]\right) + \frac{\sigma^2 f_{zz}^0}{2} E\left[\xi^2\right] dt \ ,$$

with expectations taken with respect to the approximate steady state of ξ. The first term on the right is the loss rate due to trading, which can be reexpressed in terms of u as $-\frac{1}{2}a^2 \varepsilon f_z^0 \left(bu(\beta) + cu(\gamma)\right)$. The second term is the opportunity loss rate arising because the expected return with $\xi \neq 0$ is less than the Merton optimum, $\xi = 0$. In terms of u, this loss rate is $\frac{\sigma^2 f_{zz}^0}{2} \int_\beta^\gamma \xi^2 u(\xi) d\xi$. Therefore, the inner problem is to determine β and γ so as to minimize the total rate of loss in expected utility

$$\frac{1}{2}\varepsilon a^2 f_z^0 \left(bu(\beta) + cu(\gamma)\right) - \frac{\sigma^2 f_{zz}^0}{2} \int_\beta^\gamma \xi^2 u(\xi) d\xi \ , \tag{5}$$

subject to the constraints (3), (4), and

$$\int_\beta^\gamma u(\xi) d\xi = 1 \ . \tag{6}$$

Assuming that, to leading order, the hold region is symmetric about its Merton value, that is, $\beta = -\gamma$, we differentiate (5) with respect to γ to obtain the optimal γ and the minimal expected loss in utility discussed above. These optimal parameters β and γ depend on the outer parameters z and t.

This argument in Section 2.2 is intuitive but informal. We approach the problem from a more rigorous asymptotic analysis point of view in Section 2.3. In both sections, our goal of finding the optimal $\mathcal{H}$ is a domain optimization problem similar to optimal design problems in mechanics. Such optimization problems have corresponding dual formulations, which are called the *adjoint* in design optimization. In Section 2.4 we will establish that the equations in Section 2.3 are, in fact, the adjoint of the equations in Section 2.2, thereby justifying our intuitive approach in Section 2.2.

In Section 2.3, we approach the optimization problem by generalizing the asymptotic expansion strategy of Whalley and Wilmot [6] to a general utility function. In Section 2.2, the emphasis is on u, the probability, as in Kolmogorov's forward equation; in Section 2.3, the emphasis is on the value function for the expected utility

$$f(z, \xi, t, \varepsilon) = \sup E_{z,\xi,t}\left[U(Z(T))\right] \ , \tag{7}$$

that satisfies a backward equation, the Hamilton-Jacobi-Bellman equation.

As detailed in Section 2.3, we consider the asymptotic expansion

$$f(z, \xi, t, \varepsilon) \sim f^0(z,t) + \varepsilon^{2/3} f^2(z,t) + \varepsilon^{4/3} f^4(z, \xi/\varepsilon^{1/3}, t) \ , \tag{8}$$

where $z = x + y$ represents the book value of the portfolio. The inner problem now becomes a free boundary problem for f^4, expressed as a function of the scaled imbalance $\widetilde{\xi} = \xi/\varepsilon^{1/3}$, over the as-yet-undetermined hold region $\widetilde{\xi} \in [\widetilde{\beta}, \widetilde{\gamma}]$. Specifically, we find

$$\frac{1}{2} a^2 f^4_{\widetilde{\xi}\widetilde{\xi}} + \frac{1}{2}\sigma^2 \xi^2 f^0_{zz} = K(z,t) \ , \tag{9}$$

(subscripts denote partial derivatives) with first order boundary conditions

$$0 = -b f^0_z + f^4_{\widetilde{\xi}} \ , \quad \text{when } \widetilde{\xi} = \widetilde{\beta} \quad , \quad 0 = c f^0_z + f^4_{\widetilde{\xi}} \ , \quad \text{when } \widetilde{\xi} = \widetilde{\gamma} \quad , \tag{10}$$

and second order order optimality boundary conditions

$$f^4_{\widetilde{\xi}\widetilde{\xi}} = 0 \ , \quad \text{when } \widetilde{\xi} = \widetilde{\beta} \text{ or } \widetilde{\xi} = \widetilde{\gamma}. \tag{11}$$

The diffusion related coefficient, a, in (9) is defined as before. The K in (9) is a constant (as a function of $\widetilde{\xi}$) that is determined as a solvability condition. The analysis shows that, to leading order, $\beta = -\gamma$ and also yields expressions for γ and the leading order loss in expected utility, f^2. These expressions completely agree with the expressions derived in Section 2.2.

In the expressions for γ and f^2, we will have that b and c, the buying and selling transaction coefficients, only appear as the sum $b + c$. This is the cost of a "round trip", buying then immediately selling the same share of stock, hence it corresponds to a rough model of the bid-ask spread. If we buy then sell a share of stock, even without other fees or losses, we at least pay the spread.

In Section 2.4, we show that the Lagrange multiplier variable that is adjoint to u exactly satisfies the f^4 equations (9), (10), and (11), thereby establishing the equivalence of the two approaches. The equivalence of the "u" approach from Section 2.2 and the "f^4" approach from Section 2.3 is not particularly surprising given that both methods provide the same results for γ and f^2. But in the multiple stock case, where explicit solutions for the hold region and the leading order loss in the expected utility are not yet available, we will still be able to show that the two problems are equivalent, using the duality theory.

In Section 3, we consider the problem with multiple stocks. Section 3 is structured to parallel the structure of Section 2 for single stocks as much as possible. In Section 3.1 we again recall the (now multivariate) idealized Merton solution, $m_i(z,t)$, for the optimal investment in each stock i given a total wealth z and no transaction costs. With transaction costs, the value function, which is discussed in Section 3.3, has a formal asymptotic expansion of the form (8), as was pointed out by Atkinson and Mokkhavesa [7]. The inner problem for f^4 satisfies an elliptic free boundary problem in the scaled imbalance variables $\widetilde{\xi}_i = x_i - m_i(z,t)$. This has an explicit classical solution (corresponding to the viscosity solution in [7]) for the case of uncorrelated asset prices as we discuss in Section 3.5. In Section 3.2, we use informal reasoning to determine a different elliptic free boundary value problem for the steady state probability density, $u(\xi)$. In Section 3.4, the optimization problem in terms of u from Section 3.2 is shown to be equivalent to the problem for f^4 in Section 3.3, even in cases when the solution is not available. With the approaches in Sections 3.2 and 3.3 shown to be equivalent, we apply, in Section 3.6, our knowledge of u from Section 3.2 to the uncorrelated case where optimal hold region is known from Section 3.5. By relating work of Trefethen and Williams [11], we will show that u generally blows up at two corners of the rectangular hold region in the case of two uncorrelated stocks.

For the convenience of the reader, we provide in Appendix 1 an informal derivation of the second order smooth pasting optimality conditions (see, for example, Dixit [12] and Dumas [13]) both for the one stock and multiple stock problems. These are particularly helpful in identifying the second order conditions for the value function, f. These derivations use calculus of variations arguments that are common in engineering optimal design problems.

Finally, we remark that the coefficient formula from Section 2.2, $a(z,t) = \sigma m(z,t)(1 - m_z(z,t))$, has the interesting consequence that none of the expansions derived here (and in earlier work mentioned above) are correct when $m_z = 1$, which can happen even for power law utilities. The scalings change because it takes less trading to keep close to the Merton position. Indeed, consider the effect of a small dZ. If $m_z = 1$, the Merton rebalancing calls for $dX = dZ$. But $dZ \approx dX$ because stochastic changes are of a larger order of magnitude (in small time) than deterministic changes, so, to leading order, there is no need to rebalance. We study this case in more detail in [14].

2 Analysis for Single Stock Portfolios

2.1 Merton Case (No Transaction Costs)

We begin by considering a portfolio of cash and stock where no transaction costs are incurred for buying or selling stock. If our portfolio is worth a total of z dollars at time t, we seek a trading strategy for the stock at subsequent times that maximizes the expected utility of Z, the total worth of the portfolio, at a later time T. We will let f denote the maximum expected utility; that is,

$$f(z,t) = \sup \left(E_{z,t} \left[U \left(Z(T) \right) \right] \right). \tag{12}$$

If we assume that the cash can be invested at the risk-free rate r and the stock price is lognormal with mean μ and volatility σ, we have for $X(t)$, the dollar worth of stock in our portfolio at time t, and $Y(t)$, the worth of the cash in our portfolio at time t, that

$$\begin{aligned} dX(t) &= \mu X(t)dt + \sigma X(t)dB(t) && (13)\\ dY &= rY dt, && (14) \end{aligned}$$

where $B(t)$ is a Brownian motion. Since the total worth of the portfolio is the value of the stock plus the cash,

$$Z(t) = X(t) + Y(t), \tag{15}$$

we have that

$$\begin{aligned} dZ(t) &= \left[\mu X(t) + rY(t) \right] dt + \sigma X(t)dB(t) \\ &= \left[\left(\mu - r \right) X(t) + rZ(t) \right] dt + \sigma X(t)dB(t). && (16) \end{aligned}$$

Since trading is free, our trading strategy for the stock is based solely on Z and t; that is, $X(t) = x(Z(t), t)$ for some trading strategy function x, and so the Hamilton-Jacobi-Bellman equation for $f(Z,t)$ is

$$0 = \sup_x \left\{ f_t + \left[(\mu - r)\, x + rZ \right] f_z + \frac{1}{2}\sigma^2 x^2 f_{zz} \right\}. \qquad (17)$$

We define the Merton value, $m(Z,t)$, to be the value of x that maximizes the right hand side of this equation. This is determined by setting the derivative with respect to x of the braced term in (17) equal to zero

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} \left\{ f_t + \left[(\mu - r)\, x + rZ \right] f_z + \frac{1}{2}\sigma^2 x^2 f_{zz} \right\} \Bigg|_{x=m} \\ &= (\mu - r)\, f_z + \sigma^2 m f_{zz}, \end{aligned}$$

which yields the formula for the Merton value:

$$m(Z,t) = -\frac{(\mu - r)\, f_z}{\sigma^2 f_{zz}}. \qquad (18)$$

Note that m is positive since $f_z > 0$ and $f_{zz} < 0$, which can be established using the fact that $U' > 0$ and $U'' < 0$ for all utility functions.

Next we consider the effect of adding small proportional transaction costs to this model.

2.2 Heuristic Discussion of the Transaction Costs Case

Let $L(t)$ be the dollar amount of cash spent buying stock up to time t, and $M(t)$ be the dollar worth of all stock sold up to time t. For the proportional transaction cost model with a transaction parameter ε and transaction constants b and c, we update (13) and (14), the differentials of X and Y, to incorporate L and M:

$$dX(t) = \mu X(t) dt + \sigma X(t) dB(t) + (1 - \varepsilon b) dL(t) - dM(t) \qquad (19)$$

$$dY(t) = rY(t) dt + (1 - \varepsilon c) dM(t) - dL(t). \qquad (20)$$

Notice that a unit increase in L removes a unit of Y but adds only $(1 - \varepsilon b)$ units to X; that is, there is a loss of εb dollars for every dollar spent buying stock. Similarly, selling a dollar of stock removes a dollar from X but adds only $(1 - \varepsilon c)$ dollars to Y for a loss of εc dollars.

As in the Merton analysis, we use

$$Z(t) = X(t) + Y(t), \tag{21}$$

which is the book dollar value of the portfolio at time t. If the portfolio were liquidated, it would only produce $(1-\varepsilon c)X + Y$ of cash. We now introduce the key variable ξ to denote the difference between the current stock position and the ideal Merton stock position, $m(Z,t)$; that is, we define

$$\xi(t) = X(t) - m(Z(t),t). \tag{22}$$

The analysis of the transaction case is significantly simplified by using (z,ξ), as opposed to (x,y), coordinates. Combining (19) through (22) and applying the Ito calculus short hand $(dB)^2 = dt$ gives the differential of Z,

$$dZ = \left[(\mu - r)\,(m+\xi) + rZ\right] dt + \sigma(m+\xi)dB - \varepsilon(bdL + cdM) \tag{23}$$

and the differential of ξ,

$$\begin{aligned} d\xi &= dX - m_z dZ - m_t dt - \frac{1}{2} m_{zz} (dZ)^2 \\ &= \left[\begin{array}{c} \mu\,(m+\xi) - m_t \\ -m_z\left((\mu - r)\,(m+\xi) + rz\right) \\ -\frac{1}{2} m_{zz} \sigma^2 (m+\xi)^2 \end{array} \right] dt \\ &\quad + \sigma\,(1-m_z)\,(m+\xi) dB \\ &\quad + \left(1 - \varepsilon b(1-m_z)\right) dL - \left(1-\varepsilon c m_z\right) dM. \end{aligned} \tag{24}$$

We propose a probabilistic way to estimate both the optimal trading strategy and the expected loss in maximum utility due to trading costs when the trading costs are small. As has been suggested in the introduction, after an initial immediate trade, we expect the optimal trading strategy to be a singular control that operates to keep the portfolio within a certain narrow region near the Merton portfolio. We call the inside of this narrow region the *hold region* since we do not trade when we are within it. In the present variables, this means keeping ξ, the deviation from the Merton balance, small. In particular, we assume in this section that there is a small $\gamma(z,t) > 0$ so that $dL \neq 0$ only when $\xi = -\gamma$ and $dM \neq 0$ only when $\xi = \gamma$; that is, we only trade on the boundary of the hold region, $\xi \in [-\gamma, \gamma]$. The more rigorous PDE based asymptotic expansion in the next section will validate our assumption here that, to leading order, the hold region is a symmetric interval of ξ values centered about the Merton value, $\xi = 0$.

Given that the hold region for ξ should be small and slowly varying, we expect that ξ is roughly in stochastic equilibrium within this region. That is, if we imagine Z and t as fixed, the one dimensional process $\xi(t)$ will have a probability distribution that is (approximately) an equilibrium for its stochastic differential equation with reflecting boundary conditions at $\xi = \pm\gamma$.

To determine this equilibrium distribution for small γ, we make several simplifying approximations to (24). First, we neglect the drift terms — that is, all of the dt terms — which should have a small effect on the steady state since ξ stays within a small domain. Second, we neglect ξ in $(m + \xi)dB$ because ξ should be much smaller than m. Finally, we neglect the $O(\varepsilon)$ terms. This leaves us with

$$d\xi \approx dL - dM + a dB, \tag{25}$$

where a is defined by

$$a = \sigma\,(1 - m_z)\,m. \tag{26}$$

Since L and M are nondecreasing, $dL \neq 0$ only when $\xi = -\gamma$, and $dM \neq 0$ only when $\xi = \gamma$, the differential equation corresponding to (25) for the equilibrium probablility density, $u(\xi)$, is

$$A u_{\xi\xi} = 0, \tag{27}$$

subject to the Neumann boundary conditions,

$$u_\xi = 0 \text{ at } \xi = \pm\gamma, \tag{28}$$

where A is defined by

$$A = \frac{1}{2}a^2. \tag{29}$$

The solution of this equation is the uniform probability density over the hold region:

$$u(\xi) = \frac{1}{2\gamma}.$$

Let $f(z,t)$ be the idealized Merton value function in (12). We want to know how much the optimal transaction cost strategy underperforms the Merton zero transaction cost strategy. In this section, we assume the heuristic that the maximum expected utility is affected by small transaction costs primarily through the variable Z and not through direct changes to f. We consider the effect of direct changes to f in the next sections, which will justify this heuristic. Under this heuristic, we calculate

$$df = f_t dt + f_z dZ + \frac{1}{2} f_{zz} (dZ)^2,$$

with $(dZ)^2 = \sigma^2 (m+\xi)^2 dt$. Using (23), this leads to

$$\begin{aligned} df &= \left[\begin{array}{c} f_t + f_z \left(rZ + (\mu - r)m + (\mu - r)\xi\right) \\ + f_{zz} \left(\frac{\sigma^2 m^2}{2} + \sigma^2 m \xi + \frac{\sigma^2 \xi^2}{2} \right) \end{array} \right] dt \\ & \quad + f_z \sigma (m+\xi) dB - \varepsilon f_z (b dL + c dM). \end{aligned} \tag{30}$$

Since f satisfies the Merton equation

$$f_t + \left[(\mu - r)\, m + rZ \right] f_z + \frac{1}{2} \sigma^2 m^2 f_{zz} = 0,$$

the df expression in (30) simplifies to

$$\begin{aligned} df &= \left[(\mu - r)\xi f_z + \sigma^2 m \xi f_{zz} + \frac{\sigma^2 \xi^2}{2} f_{zz} \right] dt \\ & \quad + \sigma (m+\xi) f_z dB - \varepsilon f_z (b dL + c dM). \end{aligned} \tag{31}$$

We next look at estimating the optimal $\gamma(z,t)$. We start with (31) and take the approximate equilibrium (with respect to ξ) expectation, $E[\cdot]$, and, since our uniform density function implies that $E[\xi] = 0$, we have

$$E[df] = \frac{\sigma^2 f_{zz}}{2} E\left[\xi^2\right] dt - \varepsilon f_z \left(b E[dL] + c E[dM] \right). \tag{32}$$

Recalling that $f_z > 0$ and $f_{zz} < 0$, we can see that choosing γ to minimize our transaction loss — that is, choosing γ to maximize the negative quantity $E[df]$ — requires a balance between the two terms on the right hand side of (32). The first term on the right hand side is the *opportunity cost*, which gives the loss due to the portfolio's deviation from the optimal Merton balance. Since $\rho(\xi) = \frac{1}{2\gamma}$,

$$E\left[\xi^2\right] = \frac{1}{2\gamma} \int_{-\gamma}^{\gamma} \xi^2 d\xi = \frac{\gamma^2}{3}, \tag{33}$$

so we see how the opportunity cost is reduced by reducing γ. The second term on the right hand side of (32) is the *trading cost*, which gives the loss due to trading on the boundaries $\xi = \pm\gamma$. Since this cost is proportional to the density of ξ on the boundaries, it is inversely proportional to γ and so we see how the trading cost is increased by reducing γ.

More precisely, to evaluate $E[dL]$ and $E[dM]$ in the trading cost, we start with the Ito chain rule formula:

$$E\left[d\left(\xi^2\right)\right] = 2E\left[\xi d\xi\right] + E\left[(d\xi)^2\right]. \tag{34}$$

From our assumption of approximate equilibrium, we have, essentially, that $\frac{d}{dt}E\left[\xi^2\right] = 0$ so the left hand side of (34) can be neglected. To evaluate the leading order behavior of each of the two terms on the right hand side of (34), we apply (25) to find that

$$E\left[(d\xi)^2\right] = a^2 dt$$

and

$$E\left[\xi d\xi\right] = E[\xi dL] - E[\xi dM].$$

As noted above, $dL \neq 0$ only when $\xi = -\gamma$ and $dM \neq 0$ only when $\xi = \gamma$. Therefore $E[\xi dL] = -\gamma E[dL]$ and $E[\xi dM] = \gamma E[dM]$. Moreover, symmetry requires that $E[dL] = E[dM]$. So, from (29) and (34) we have

$$E[dL] = E[dM] = \frac{A}{2\gamma} dt. \tag{35}$$

Now we can evaluate $E[df]$ by substituting (33) and (35) into (31), which yields

$$\frac{E[df]}{dt} = \frac{\sigma^2 f_{zz}\gamma^2}{6} - \varepsilon f_z (b+c)\frac{A}{2\gamma}. \tag{36}$$

Maximizing $\frac{E[df]}{dt}$ over γ leads to the optimal width of the hold region:

$$\gamma = \varepsilon^{\frac{1}{3}} \left(-\frac{3}{2}\frac{A(b+c)f_z}{\sigma^2 f_{zz}}\right)^{\frac{1}{3}}. \tag{37}$$

Inserting (37) back into (36) yields the optimal loss rate:

$$\frac{E[df]}{dt} = \varepsilon^{\frac{2}{3}} \frac{\left(\sigma^2 f_{zz}\right)^{\frac{1}{3}}}{2} \left(\frac{3}{2}A(b+c)f_z\right)^{\frac{2}{3}}. \tag{38}$$

These final two formulas, (37) and (38), for γ and the optimal loss rate will be corroborated by the more rigorous free boundary PDE asymptotic expansion in the next section.

2.3 Asymptotic Expansion for the Transaction Costs Case

The fact that (37) along with previous works ([3], [4], [6], [10]) show that the width of the hold region for ξ is $O\left(\varepsilon^{\frac{1}{3}}\right)$ suggests that our expansion for f, the value function, should be in powers of $\varepsilon^{\frac{1}{3}}$. It also suggests rescaling ξ so that ξ from the previous section will now be replaced by $\varepsilon^{\frac{1}{3}}\xi$ in this section (although ξ will revert back to Section 2.2's scaling in Section 2.4). Therefore, the differentials for dZ and $d\xi$ in (23) and (24) now become

$$\begin{aligned} dZ &= \left[(\mu-r)\left(m+\varepsilon^{\frac{1}{3}}\xi\right)+rZ\right]dt+\left[(m+\varepsilon^{\frac{1}{3}}\xi)\sigma\right]dB \\ &\quad -\varepsilon(bdL+cdM) \end{aligned} \tag{39}$$

$$d\xi = \frac{1}{\varepsilon^{\frac{1}{3}}}\left\{\begin{array}{c} \left[\begin{array}{c} \mu\left(m+\varepsilon^{\frac{1}{3}}\xi\right)-m_t \\ -m_z\left((\mu-r)\left(m+\varepsilon^{\frac{1}{3}}\xi\right)+rZ\right) \\ -\frac{1}{2}m_{zz}\sigma^2(m+\varepsilon^{\frac{1}{3}}\xi)^2 \end{array}\right]dt \\ +\left[(1-m_z)\left(m+\varepsilon^{\frac{1}{3}}\xi\right)\sigma\right]dB \\ +\left(1-\varepsilon b(1-m_z)\right)dL-\left(1-\varepsilon c m_z\right)dM \end{array}\right\}. \tag{40}$$

Given this, on the interior of the hold region where $dL = dM = 0$, the Hamilton-Jacobi-Bellman equation for

$$f(z,\xi,t)=\sup\left\{E_{z,\xi,t}\left[U\left(Z(T)\right)\right]\right\}$$

is

$$\begin{aligned} 0 &= f_t+\left[(\mu-r)\left(m+\varepsilon^{\frac{1}{3}}\xi\right)+rZ\right]f_z \\ &\quad +\frac{1}{\varepsilon^{\frac{1}{3}}}\left[\begin{array}{c} \mu\left(m+\varepsilon^{\frac{1}{3}}\xi\right)-m_t-m_z\left((\mu-r)\left(m+\varepsilon^{\frac{1}{3}}\xi\right)+rZ\right) \\ -\frac{1}{2}m_{zz}\sigma^2(m+\varepsilon^{\frac{1}{3}}\xi)^2 \end{array}\right]f_\xi \\ &\quad +\frac{1}{2}\frac{1}{\varepsilon^{\frac{2}{3}}}\left[(1-m_z)^2\sigma^2\left(m+\varepsilon^{\frac{1}{3}}\xi\right)^2\right]f_{\xi\xi} \\ &\quad +\frac{1}{\varepsilon^{\frac{1}{3}}}\left[(1-m_z)\sigma^2(m+\varepsilon^{\frac{1}{3}}\xi)^2\right]f_{\xi z} \\ &\quad +\frac{1}{2}\left[\sigma^2(m+\varepsilon^{\frac{1}{3}}\xi)^2\right]f_{zz}. \end{aligned} \tag{41}$$

On the boundaries of the hold region — where trading occurs — the Hamilton-Jacobi-Bellman equation is dominated by the dL or dM terms since, on their respective boundaries, $E[dL] = E[dM] = O\left(\sqrt{dt}\right)$ (see McKean [15]). Specifically, on the boundary of the hold region where we are buying stock, the highest order term is

$$0 = \left[-\varepsilon b f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_\xi - \varepsilon^{\frac{2}{3}} b(1-m_z) f_\xi\right] dL,$$

so the coefficient of dL must be zero, and we get the following first order derivative condition on this boundary:

$$0 = -\varepsilon b f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_\xi - \varepsilon^{\frac{2}{3}} b(1-m_z) f_\xi. \tag{42}$$

Similarly, on the boundary where we are selling stock,

$$0 = \left[\varepsilon a f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_\xi - \varepsilon^{\frac{2}{3}} c m_z f_\xi\right] dM,$$

so here we have the first order boundary condition

$$0 = \varepsilon a f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_\xi - \varepsilon^{\frac{2}{3}} c m_z f_\xi. \tag{43}$$

As explained in Appendix 1, the fact that the boundaries of the hold region are chosen to optimize f leads to second order boundary conditions that can be obtained by taking the derivatives of both of the first order boundary conditions, (42) and (43), with respect to ξ. So on the boundary where we buy stock,

$$0 = -\varepsilon b f_{z\xi} + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi\xi} - \varepsilon^{\frac{2}{3}} b(1-m_z) f_{\xi\xi}, \tag{44}$$

and on the boundary where we sell stock,

$$0 = \varepsilon a f_{z\xi} + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi\xi} - \varepsilon^{\frac{2}{3}} c m_z f_{\xi\xi}. \tag{45}$$

Note in the first order boundary conditions, (42) and (43), that the f_z term is $O\left(\varepsilon^{\frac{4}{3}}\right)$ larger than the highest order term for f_ξ. This implies that ξ affects the solution for f only at the $O\left(\varepsilon^{\frac{4}{3}}\right)$ level, and so we consider an expansion for f of the form

$$f = f^o(z,t) + \varepsilon^{\frac{1}{3}} f^1(z,t) + \varepsilon^{\frac{2}{3}} f^2(z,t) + \varepsilon f^3(z,t) + \varepsilon^{\frac{4}{3}} f^4(z,\xi,t) + \ldots \tag{46}$$

Now we begin to set up the matched asymptotic expansion by first collecting terms in (41) with identical powers of $\varepsilon^{\frac{1}{3}}$:

$$
\begin{aligned}
0 \;=\; & \frac{1}{\varepsilon^{\frac{2}{3}}}\left[\frac{1}{2}\left(1-m_z\right)^2\sigma^2 m^2 f_{\xi\xi}\right] \\
& +\frac{1}{\varepsilon^{\frac{1}{3}}}\left[\begin{array}{c} \{\mu m - m_t - m_z\left(\left(\mu-r\right)m + rz\right) \\ -\frac{1}{2}m_{zz}\sigma^2 m^2\}f_\xi \\ +\left(1-m_z\right)^2\sigma^2 m\xi f_{\xi\xi} \\ +\left(1-m_z\right)\sigma^2 m^2 f_{\xi z} \end{array}\right] \\
& +\varepsilon^0\left[\begin{array}{c} f_t + \{\left(\mu-r\right)m + rz\}f_z \\ +\{\mu\xi - m_z\left(\left(\mu-r\right)\xi\right) - \frac{\partial^2 m}{\partial z^2}\sigma^2 m\xi\}f_\xi \\ +\frac{1}{2}\left(1-m_z\right)^2\sigma^2\xi^2 f_{\xi\xi} + 2\left(1-m_z\right)\sigma^2 m\xi f_{\xi z} + \frac{1}{2}\sigma^2 m^2 f_{zz} \end{array}\right] \\
& +\varepsilon^{\frac{1}{3}}\left[\begin{array}{c} \left(\mu-r\right)\xi f_z - \frac{1}{2}m_{zz}\sigma^2\xi^2 f_\xi \\ +\left(1-m_z\right)\sigma^2\xi^2 f_{\xi z} + \sigma^2 m\xi f_{zz} \end{array}\right] \\
& +\varepsilon^{\frac{2}{3}}\left[\frac{1}{2}\sigma^2\xi^2 f_{zz}\right]
\end{aligned} \tag{47}
$$

and then, after inserting (46), the expansion for f, into (47), we again collect terms with identical powers of $\varepsilon^{\frac{1}{3}}$, each of which is required to be zero for the expansion to be satisfied. The order ε^0 terms from this expansion lead, of course, to the Merton (no transaction cost) equation for f^0:

$$
0 = f_t^0 + \{\left(\mu-r\right)m + rz\}f_z^0 + \frac{1}{2}\sigma^2 m^2 f_{zz}^0.
$$

The order $\varepsilon^{\frac{1}{3}}$ terms contain both f^0 and f^1 parts:

$$
\begin{aligned}
0 \;=\; & \xi\left[\left(\mu-r\right)f_z^0 + \sigma^2 m f_{zz}^0\right] \\
& +\left[f_t^1 + \{\left(\mu-r\right)m + rz\}f_z^1 + \frac{1}{2}\sigma^2 m^2 f_{zz}^1\right].
\end{aligned} \tag{48}
$$

Since, from (18) we already know that the Merton values are

$$
m = -\frac{\left(\mu-r\right)f_z^0}{\sigma^2 f_{zz}^0},
$$

we see, by substitution, that the f^0 terms in (48) cancel each other, which leaves the Merton equation for f^1, but since the final condition at time T for

f^1 is $f^1(z,T)=0$, we must have, by uniqueness, that $f^1(z,t)=0$. Setting $f^1=0$, we now collect the order $\varepsilon^{\frac{2}{3}}$ terms:

$$\begin{aligned} 0 &= \frac{1}{2}(1-m_z)^2\sigma^2 m^2 f^4_{\xi\xi} + \\ &\quad f^2_t + \{(\mu-r)m+rz\}f^2_z + \frac{1}{2}\sigma^2 m^2 f^2_{zz} + \frac{1}{2}\sigma^2\xi^2 f^0_{zz}. \end{aligned} \tag{49}$$

Since only f^4 depends on ξ, we see that (49) is an ODE in the ξ variable for f^4 of the form

$$K = Af^4_{\xi\xi} + \frac{1}{2}f^0_{zz}\sigma^2\xi^2, \tag{50}$$

where A (defined in (29)) and K, the differential expression for f^2:

$$K = f^2_t + \{(\mu-r)m+rz\}f^2_z + \frac{1}{2}\sigma^2 m^2 f^2_{zz},$$

both behave like constants since they have no ξ dependence. Integrating (50) twice with respect to ξ yields

$$\frac{K}{2}\xi^2 + C\xi + D = Af^4 + \frac{1}{24}f^0_{zz}\sigma^2\xi^4 \tag{51}$$

where C and D are constants of integration.

Now we consider the boundary conditions at the order $\varepsilon^{\frac{2}{3}}$ level. We define $\xi=\beta(z,t)$ to be the boundary of the hold region where stock is bought and $\xi=\gamma(z,t)$ to be the boundary of the hold region where stock is sold. From (42), we have that the first derivative boundary condition on β is

$$0 = -bf^0_z + f^4_\xi \tag{52}$$

and from (43), we have that the first derivative boundary condition on γ is

$$0 = cf^0_z + f^4_\xi. \tag{53}$$

From (44) and (45), we have that the second derivative boundary condition on both β and γ is

$$0 = f^4_{\xi\xi}. \tag{54}$$

Applying this second derivative condition to (50) yields

$$\frac{2K}{f^0_{zz}\sigma^2} = \beta^2 = \gamma^2 \tag{55}$$

so, since $\gamma > 0$ and $\beta < 0$, we have that $\gamma = -\beta$; that is, at the first order of approximation, the buy and the sell boundary are equidistant from the Merton value, $\xi = 0$, which validates our assumption of a symmetric hold region in the previous section. Next we apply the two first derivative conditions to (51) at $\xi = -\gamma$ and at $\xi = \gamma$ which gives

$$\begin{aligned} 0 &= -bAf_z^0 - K\gamma + C + \frac{1}{6} f_{zz}^0 \sigma^2 \gamma^3 && (56) \\ 0 &= cAf_z^0 + K\gamma + C - \frac{1}{6} f_{zz}^0 \sigma^2 \gamma^3. && (57) \end{aligned}$$

Adding (56) and (57) gives one of the constants of integration in (51):

$$C = \frac{1}{2} A(b-c) f_z^0.$$

More importantly, subtracting (56) from (57) gives

$$0 = -(b+c) f_z^0 - \frac{2K}{A}\gamma + \frac{1}{3A} f_{zz}^0 \sigma^2 \gamma^3$$

and substituting for K using (55) allows us to isolate γ, giving us the nature of the hold region (to the first order of approximation):

$$\gamma(z,t) = \left(-\frac{3}{2} \frac{A(b+c) f_z^0}{\sigma^2 f_{zz}^0} \right)^{\frac{1}{3}}. \tag{58}$$

Further, inserting this expression for γ into (55) yields a nonhomogeneous PDE for f^2, the first order approximation to the loss in the maximum expected utility due to transaction costs:

$$\begin{aligned} & f_t^2 + \{(\mu - r)\, m + rz\} f_z^2 + \frac{1}{2} \sigma^2 m^2 f_{zz}^2 \\ = \; & \frac{(\sigma^2 f_{zz}^0)^{\frac{1}{3}}}{2} \left(\frac{3}{2} A(b+c) f_z^0 \right)^{\frac{2}{3}} && (59) \end{aligned}$$

where the final condition at time T is

$$f^2(z,T) = 0.$$

We note that the expression for γ in (58) agrees with (37), the expression for γ from the previous section. Also, since $E[df]$ from the previous section equals (to leading order) $\varepsilon^{\frac{2}{3}} f^2$ from this section and $\frac{df^2}{dt} = f_t^2 + f_z^2 (dZ) + \frac{1}{2} f_{zz}^2 (dZ)^2$, which, to leading order, equals $f_t^2 + \{(\mu - r)\, m + rz\} f_z^2 + \frac{1}{2}\sigma^2 m^2 f_{zz}^2$, we see that the expression for the loss in maximum expected utility (59) also agrees with (38), the expression for the loss in maximum expected utility from the previous section.

2.4 Lagrange Multiplier Perspective for the Maximum Expected Utility

We have just shown that the heuristic perspective from Section 2.2 and the more rigorous asymptotic expansions from Section 2.3 lead to the same expressions for both γ, the optimal hold region and f_2, the leading order term for the loss in maximum expected utility. In this section we establish a deeper connection between these two sections' approaches. The connection established here will be extended to the case of multiple stocks in Section 3.4, even though the formulas for γ and f_2 will not generally be extendable. Section 2.2 essentially looks at the probability density of the portfolio whereas Section 2.3 essentially looks at the maximum expected utility. It is well known that integration by parts can be used to show that the forwards equation for the probabilty density is dual to the backwards equation for the maximum expected utility. Similarly, in this section we use integration by parts to show that the Lagrange multiplier function that is dual to the constraints on the probability in the constrained optimization described in Section 2.2 corresponds to f_4, the leading order term depending on ξ in the maximum expected utility from Section 2.3. This Lagrange multiplier function and f_4 correspond in the sense that they both solve the same equations, namely, the differential equation (49) in the interior of the hold region, the two first order boundary conditions (52), (53), and the two second order boundary conditions (54). Further, since these equations only involve derivatives with respect to ξ, we can see that this Lagrange multiplier function also corresponds — in the same sense of satisfying these equations — to the first five terms in the expansion for f, specifically the f_0 through f_4 terms in expansion (46) of Section 2.3.

The equations satisfied by the Lagrange multiplier function will be derived using the method shown in Appendix 2. In Appendix 2, we detail how to optimize a functional, in our case here the maximum expected utility, subject to constraints. Further, both the functional and the constraints depend on two functions, denoted by u and v in Appendix 2, where u is completely determined by v through the constraints. In our case here, the v of Appendix 2 is γ, which defines $\mathcal{H}$, the hold region $\xi \in (-\gamma, \gamma)$. The u of Appendix 2 is $u(\xi)$ here, the invariant probability density, which, once we know γ, is uniquely determined through the following three constraints: The first constraint is (27), the differential equation constraint

$$Au_{\xi\xi} = 0 \quad \text{for } \xi \in \mathcal{H} \; , \tag{60}$$

where, recall from (26) and (29), A is defined by

$$A = \frac{\sigma^2(1-m_z)^2 m^2}{2} \; . \tag{61}$$

Since the case where $A = 0$ is highly unlikely to occur, we will assume that $A > 0$ for the remainder of this paper.[1] The second constraint is (28), the Neumann boundary constraint

$$\mp A u_\xi(\pm\gamma) = 0 \; . \tag{62}$$

In (62) we multiplied $u_\xi = 0$ by A or $-A$ because it yields more simple results that generalize more clearly to the multi-stock case later. The third constraint is the normalization constraint for probability densities

$$\int_{\mathcal{H}} u(\xi) d\xi = \int_{-\gamma}^{\gamma} u(\xi) d\xi = 1 \; . \tag{63}$$

Although we know that the solution to these three constraints must be $u(\xi) = \frac{1}{2\gamma}$, we will not use this fact directly here, since we want our method to be extendable in Section 3 to the multi-stock portfolio case where, as we will establish, the density is not generally constant in the hold region.

In this spirit, we must also express the functional to be optimized, $E[df]$, the loss in maximum expected utility, without using the equation $u(\xi) = \frac{1}{2\gamma}$ directly. Recall from (32) that

$$E[df] = \frac{\sigma^2 f_{zz}}{2} E\left[\xi^2\right] dt - \varepsilon f_z \left(bE[dL] + cE[dM]\right) \tag{64}$$

[1] In [14], however, we show that when a varible amount of option is added to the portfolio, we can use the option to remain close to the analog of the $A = 0$ case, which reduces the order of the transaction costs from $O\left(\varepsilon^{\frac{2}{3}}\right)$, as we have shown is the case here in equation (38), to $O\left(\varepsilon^{\frac{6}{7}}\right)$.

where $E\left[\xi^2\right] = \int_{-\gamma}^{\gamma} \xi^2 u(\xi) d\xi$. Determining $E[dL]$ and $E[dM]$ in terms of the probability density is easier to understand without assuming equilibrium; that is, using

$$u_t = A u_{\xi\xi} \quad \text{for } \xi \in \mathcal{H} \tag{65}$$

for the moment in place of (60). Given this equation, we can compute $\frac{d}{dt}E[\phi(\xi(t))]$ where ϕ is an arbitrary smooth function, in two ways. Applying (65) and integrating by parts, we have

$$\begin{aligned} \frac{d}{dt}E\left[\phi(\xi(t))\right] &= \frac{d}{dt}\int_{\mathcal{H}} \phi(\xi) u(\xi,t) d\xi \\ &= A\left\{\int_{\mathcal{H}} \phi_{\xi\xi} u d\xi + \phi u_\xi \Big|_{\partial\mathcal{H}} - \phi_\xi u \Big|_{\partial\mathcal{H}}\right\} \end{aligned}$$

where the middle term of the last line vanishes by the Neumann boundary constraint, (62). On the other hand, applying Ito calculus using (25), together with the fact that $dL = 0$ unless $\xi(t) = -\gamma$ and that $dM = 0$ unless $\xi(t) = \gamma$, yields

$$dE\left[\phi(\xi(t))\right] = \frac{1}{2}E\left[\phi_{\xi\xi}(d\xi)^2\right] + \phi_\xi(-\gamma)E[dL] - \phi_\xi(\gamma)E[dM]\ .$$

Since ϕ is arbitrary, reconciling these two expressions gives

$$\begin{aligned} E[dL] &= Au(-\gamma,t)dt \\ E[dM] &= Au(\gamma,t)dt\ . \end{aligned}$$

If we again assume equilibrium so u no longer depends on time, we have from (64) that we want to optimize

$$f_{zz}\frac{\sigma^2}{2}\int_{\xi\in\mathcal{H}} \xi^2 u(\xi) d\xi \ - \ \varepsilon f_z A\left(bu(-\gamma) + cu(\gamma)\right) \tag{66}$$

subject to the constraints (60), (62), and (63) on u. This completes the formulation of our constrained optimization problem.

Now we are ready to follow the strategy outlined in Appendix 2 to derive optimality conditions using Lagrange multipliers for each of the three constraints. Corresponding to the differential equation constraint, (60), we have the Lagrange multiplier function $\lambda(\xi)$, which is defined for $\xi \in \mathcal{H}$. Corresponding to the Neumann boundary constraint, (62), we have the Lagrange multiplier function $\mu(\xi)$, which is defined for the two endpoints, $\xi \in \partial\mathcal{H}$, so,

μ has just two values which we denote as $\mu_\pm = \mu(\pm\gamma)$. Corresponding to the normalization constraint, (63), we have the Lagrange multiplier scalar ν. Given this, the expression (146) in Appendix 2 takes the form

$$f_{zz}\frac{\sigma^2}{2}\int_{\mathcal{H}}\xi^2 u(\xi)d\xi - \varepsilon f_z A\left(bu(-\gamma)+cu(\gamma)\right)$$
$$-A\int_{\mathcal{H}}\lambda(\xi)u_{\xi\xi}(\xi)d\xi$$
$$+A\mu_+ u_\xi(\gamma) - A\mu_- u_\xi(-\gamma) - \nu\left(\int_{\mathcal{H}}u(\xi)d\xi - 1\right) ,$$

which, after applying integration by parts twice to the $u_{\xi\xi}$ term, can be written in the more useful form

$$\begin{aligned}\int_{\mathcal{H}}&\left(-A\lambda_{\xi\xi}(\xi) + f_{zz}\frac{\sigma^2}{2}\xi^2 - \nu\right)u(\xi)d\xi \\ &+A\left(-\lambda_\xi(-\gamma) - \varepsilon f_z b\right)u(-\gamma) \\ &+A\left(\lambda_\xi(\gamma) - \varepsilon f_z c\right)u(\gamma) \\ &+A\left(\lambda(-\gamma) - \mu_-\right)u_\xi(-\gamma) \\ &-A\left(\lambda(\gamma) - \mu_+\right)u_\xi(\gamma) \\ &+\nu \ .\end{aligned} \tag{67}$$

With this form, it is now easy to perturb u in (67) to obtain the Lagrange multiplier equation (147) from Appendix 2, which takes the form

$$\begin{aligned}0 = \int_{\mathcal{H}}&\left(-A\lambda_{\xi\xi}(\xi) + f_{zz}\frac{\sigma^2}{2}\xi^2 - \nu\right)\dot{u}(\xi)d\xi \\ &+A\left(-\lambda_\xi(-\gamma) - \varepsilon f_z b\right)\dot{u}(-\gamma) \\ &+A\left(\lambda_\xi(\gamma) - \varepsilon f_z c\right)\dot{u}(\gamma) \\ &+A\left(\lambda(-\gamma) - \mu_-\right)\dot{u}_\xi(-\gamma) \\ &-A\left(\lambda(\gamma) - \mu_+\right)\dot{u}_\xi(\gamma) \ .\end{aligned} \tag{68}$$

Since (68) holds for all arbitrary infinitesimal perturbations $\dot{u}$, we must have that inside $\mathcal{H}$,

$$A\lambda_{\xi\xi}(\xi) = f_{zz}\frac{\sigma^2}{2}\xi^2 - \nu \ , \tag{69}$$

whereas on the boundary of $\mathcal{H}$, since $A > 0$ has been assumed, we have the following Neumann conditions for λ

$$\lambda_\xi(-\gamma) = -\varepsilon b f_z \tag{70}$$
$$\lambda_\xi(\gamma) = \varepsilon c f_z \tag{71}$$

and Dirichlet conditions for $\mu_\pm$

$$\mu_\pm = \lambda(\pm\gamma) \; . \tag{72}$$

Next we perturb the boundary γ in (67) to obtain the optimality equation (148) from Appendix 2. Due to (69), (70), (71), and (72), the terms in the optimalilty equation created by the effect that peturbating γ has on $\mathcal{H}$, $u(\pm\gamma)$, and $u_\xi(\pm\gamma)$ in (67) equal zero. The nonzero terms in the optimality equation result from the effect that perturbing γ has on $\lambda(\pm\gamma)$ and $\lambda_\xi(\pm\gamma)$ in (67), so the equation takes the form

$$0 = \left(\lambda_{\xi\xi}(-\gamma)u(-\gamma) + \lambda_{\xi\xi}(\gamma)u(\gamma) - \lambda(-\gamma)u_\xi(-\gamma) + \lambda(\gamma)u_\xi(\gamma)\right)\dot{\gamma} \; . \tag{73}$$

The $u_\xi(\pm\gamma)$ terms in (73) equal zero by the Neumann condition (62), and therefore since $\dot{\gamma}$ is arbitrary and $u(\pm\gamma) > 0$, the optimality equation reduces to the two second order boundary conditions

$$\lambda_{\xi\xi}(\pm\gamma) = 0. \tag{74}$$

Further, by substituting (69) evaluated at $\pm\gamma$ into (74), we also obtain ν:

$$\nu = f_{zz}\frac{\sigma^2}{2}\gamma^2 \; . \tag{75}$$

Comparing the differential equation (69) with the differential equation (49), the Neumann conditions (70) and (71) with the Neumann conditions (52) and (53) and the second order optimality conditions (74) with the second order optimality conditions (54) (and recalling that ξ in this section and Section 2.2 was scaled to equal $\varepsilon^{\frac{1}{3}}\xi$ in Section 2.3), we see that we can identify $\lambda(\xi)$ with

$$-\varepsilon^{\frac{4}{3}} f^4(z, \varepsilon^{-1/3}\xi, t) \; .$$

Further, since f_0, f_1, f_2, and f_3 have no dependence on ξ and since the differential equation, the Neumann conditions, and the second order optimality conditions only involve terms with the derivative with respect to ξ, we see that we can also identify $\lambda(\xi)$ with

$$-\left[f^0(z,t) + \varepsilon^{\frac{1}{3}} f^1(z,t) + \varepsilon^{\frac{2}{3}} f^2(z,t) + \varepsilon f^3(z,t) + \varepsilon^{\frac{4}{3}} f^4(z, \varepsilon^{-1/3}\xi, t)\right] \; ,$$

which establishes the equivalence between the two approaches.

3 Analysis for Multiple Stock Portfolios

We now consider portfolios with n stocks— as opposed to a single stock — and cash. **Important summation convention:** Throughout this section we employ the following slightly nonstandard summation convention: Unless otherwise stated, for any term on the right hand side of an equation, we sum from 1 to n over all indices (i, j, k, l, m) that appear in the term unless the index also appears on the left hand side of the equation (in which case we do not sum over that index).

3.1 Merton version (No Transaction Costs)

For the Merton (no transaction costs) problem we again wish to determine $f(z,t) = \max\{E_{z,t}[U(Z(T))]\}$. Paralleling our previous notation, we let $X_i(t)$ represent the dollar worth of stock i in our portfolio at time t, and let the price of stock i be lognormal with mean μ_i. The volatilities of the n stocks are now related by the volatility matrix, σ_{ij}. As before, $Y(t)$ is the worth of the cash in the portfolio, which can be invested at the risk-free rate r, so

$$\begin{aligned} dX_i(t) &= \mu_i X_i(t)dt + X_i(t)\sigma_{ij}dB_j \\ dY(t) &= rY(t)dt, \end{aligned}$$

where each of the B_i are independent Brownian motions. The total worth of the portfolio, $Z(t)$, is

$$Z(t) = X_i(t) + Y(t)$$

(note the summation convention implies that we sum over all the X_i here), and so

$$dZ = (\mu_i X_i + rY)dt + X_i\sigma_{ij}dB_j.$$

As before, our trading strategy, x_i, for each stock is based solely on Z and t; that is, $X_i(t) = x_i(Z(t), t)$. Therefore the Hamilton-Jacobi-Bellman equation for f is

$$0 = \sup_{x_1,\dots x_n}\left\{f_t + [(\mu_i - r)x_i + rZ]f_z + \frac{1}{2}[x_i\sigma_{ik}\sigma_{jk}x_j]f_{zz}\right\}, \tag{76}$$

and the Merton values, m_i, which we define as the values of x_i that maximize the right hand side of this equation, are given by

$$m_i(z,t) = -\frac{\sigma_{ji}^{-1}\sigma_{jk}^{-1}(\mu_k - r)f_z}{f_{zz}}. \tag{77}$$

3.2 Heuristic Discussion of the Transaction Costs Case

Let $L_i(t)$ be the dollar amount of cash spent buying stock i up to time t, and $M_i(t)$ be the dollar worth of all stock i sold up to time t. For the transaction cost model with a transaction parameter ε and transaction constants b_i and c_i, the differentials of X_i and Y are now

$$\begin{aligned} dX_i(t) &= \mu_i X_i(t)dt + X_i(t)\sigma_{ij}dB_j + (1-\varepsilon b_i)dL_i(t) - dM_i(t) \quad (78) \\ dY(t) &= rY(t)dt + (1-\varepsilon c_i)dM_i(t) - dL_i(t). \quad (79) \end{aligned}$$

As before, we employ the variable ξ_i to denote the difference between X_i, the current worth of the portfolio's position in stock i, and $m_i(Z,t)$, the worth of the portfolio's ideal Merton position in stock i:

$$\xi_i(t) = X_i(t) - m_i(Z(t),t). \quad (80)$$

Also, as before, we use (z, ξ_i), as opposed to (x_i, y), coordinates. Combining (78) through (80) and applying the Ito calculus short hand

$$dB_i dB_j = \begin{cases} dt & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

gives the differential of Z :

$$dZ = \left[(\mu_i - r)(m_i + \xi_i) + rZ\right]dt + \left[(m_i+\xi_i)\sigma_{ij}\right]dB_j - \varepsilon\left(b_i dL_i + c_i dM_i\right) \quad (81)$$

and the differential of ξ_i :

$$\begin{aligned} d\xi_i &= dX_i - (m_i)_t\,dt - (m_i)_z\,dZ - \frac{1}{2}(m_i)_{zz}(dZ)^2 \\ &= \begin{bmatrix} \mu_i(m_i+\xi_i) - (m_i)_t \\ -(m_i)_z\left((\mu_j - r)(m_j+\xi_j) + rz\right) \\ -\frac{1}{2}(m_i)_{zz}(m_j+\xi_j)\sigma_{jk}\sigma_{lk}(m_l+\xi_l) \end{bmatrix} dt \\ &\quad + \left[(m_i+\xi_i)\sigma_{ik} - (m_i)_z(m_j+\xi_j)\sigma_{jk}\right]dB_k \\ &\quad + (1-\varepsilon b_i)\,dL_i + \varepsilon(m_i)_z\,b_j dL_j - dM_i + \varepsilon(m_i)_z\,c_j dM_j. \quad (82) \end{aligned}$$

We expect that the optimal trading strategy will again require a small hold region, $\mathcal{H}$, near the Merton portfolio. When the portfolio is inside $\mathcal{H}$ we hold, that is, do not trade, but if we hit $\partial\mathcal{H}$, the boundary of $\mathcal{H}$, we trade so that the portfolio never leaves $\mathcal{H}$. Given that the hold region for all of the ξ_i

is small and slowly varying, we can think of Z and t as fixed again, so the hold region can be pictured as a small region in $\mathbf{R}^n$. Further, the $(n-1)$-dimensional boundary of the hold region can be partitioned into $2n$ regions: β_i, where we buy stock i, and γ_i, where we sell stock i. (See Figure 1.)

As in the single variable case, we want to know how much our finite transaction cost optimal strategy underperforms the Merton zero transaction cost strategy. Therefore, we calculate

$$df = f_t dt + f_z dZ + \frac{1}{2} f_{zz} (dZ)^2,$$

and, with dZ from (81) and

$$(dZ)^2 = (m_j + \xi_j)\,\sigma_{ji} dB_i\,(m_l + \xi_l)\,\sigma_{lk} dB_k = (m_j + \xi_j)\,\sigma_{jk}\sigma_{lk}\,(m_l + \xi_l)\,dt,$$

this leads to

$$\begin{aligned} df &= \left[\begin{array}{c} f_t + f_z\left(rZ + (\mu_i - r)\,m_i + (\mu_i - r)\,\xi_i\right) \\ +f_{zz}\left(\frac{m_j\sigma_{jk}\sigma_{lk}m_l}{2} + \sigma^2 m_j \sigma_{jk}\sigma_{lk}\xi_l + \frac{\xi_j\sigma_{jk}\sigma_{lk}\xi_l}{2}\right) \end{array} \right] dt \\ &\quad + f_z\left[(m_i + \xi_i)\sigma_{ij}\right] dB_j - \varepsilon f_z\left(b_i dL_i + c_i dM_i\right). \end{aligned} \tag{83}$$

For small ε, we expect that $E[\xi_i] = 0$ approximately holds to reduce the opportunity cost. Using this and the fact that f satisfies the Merton equation,

$$0 = f_t + \left[(\mu_i - r)\,m_i + rZ\right] f_z + \frac{1}{2}\left[m_i \sigma_{ik}\sigma_{jk} m_j\right] f_{zz},$$

allows us to simplify the df expression in (83) to

$$E[df] = \frac{f_{zz}}{2} E\left[\xi_i \sigma_{ik}\sigma_{jk}\xi_j\right] dt - \varepsilon f_z\left(b_i E\left[dL_i\right] + c_i E\left[dM_i\right]\right). \tag{84}$$

As before, the first term on the right hand side of (84) is the *opportunity cost*, which gives the loss due to the portfolio's deviation from the optimal Merton balance, and the second term on the right hand side is the *trading cost*, which gives the loss due to trading on the boundary of the hold region.

The analysis of the trading cost term with multiple stocks requires a transformation so that we can eventually apply the single variable analysis. Consider a generic point, which we will call ξ_0, on β_i, the boundary where we buy stock i. We are interested in the expected ΔL_i due to contact with β_i near this point over very small time intervals, Δt. Because Δt is small,

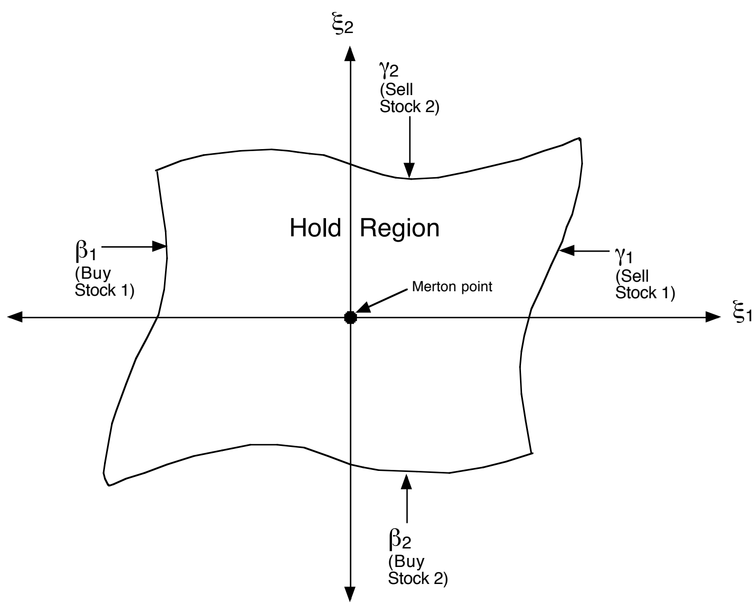

Figure 1: *The Hold Region (also called the No Transaction Region) centered about the Merton portfolio.*

contact with β_i near ξ_0 will generally be from points that are located in the hold region very near ξ_0 at the beginning of the time interval; that is, the relevant dynamics are local near ξ_0. Given this, if we assume that β_i is differentiable at ξ_0 and that the density is continuous at ξ_0, we can model ΔL_i at ξ_0 by approximating the hold region as the half space bounded by the ($n-1$ dimensional) plane tangent to β_i at ξ_0 (where ν_l will denote the components of the vector normal to this plane pointing out of the hold region) and by approximating the density to equal the constant $u(\xi_0)$ throughout the hold region.

Now we look at the differential of each of the ξ_l in (82) — note that we have shifted the index to l so that it is not confused with i, the index of the stock being bought. We make the same simplifying assumptions as we had in the single variable case: specifically, we neglect the drift terms (i.e., all of the dt terms) because the hold region is small, we neglect ξ_j in $(m_j + \xi_j)$ because ξ_j should be much smaller than m_j, and we neglect the $O(\varepsilon)$ terms, which leaves us with

$$d\xi_l \approx dL_l - dM_l + a_{lk} dB_k. \tag{85}$$

where

$$a_{lk} = m_l \sigma_{lk} - (m_l)_z \, m_j \sigma_{jk}. \tag{86}$$

Now we multiply (85) by $-\nu_l$, sum over l, and consider each resulting term. Since we are only interested in the motion of points over small time intervals near ξ_0 on β_i, all dL_l where $l \neq i$ are negligible and all dM_l are negligible, and therefore

$$-\nu_i dL_i = -\nu_l \left(dL_l - dM_l\right)$$

Now define η to be the distance from a generic point, ξ, in the hold region to the tangent plane for β_i at ξ_0. That is,

$$\eta = -\nu_l \left(\xi_l - (\xi_0)_l\right)$$

and therefore,

$$d\eta = -\nu_l d\xi_l.$$

Finally, by the joint probability density function for the normal distribution, we have the property

$$\hat{a} dB = -\nu_l a_{lk} dB_k$$

where the scalar $\widehat{a}$ is defined by

$$\widehat{a}^2 = \nu_j a_{jk} a_{lk} \nu_l$$

and B is a scalar Brownian motion. Combining these statements yields our desired differential form, the simple *scalar* expression

$$d\eta = -\nu_i dL_i + \widehat{a} dB \tag{87}$$

(where we do not sum over i). We have already analyzed these scalar dynamics, so we know from (35) that

$$E[dL_i] = -\frac{u(\xi_0)}{2\nu_i} \widehat{a}^2 dt.$$

In other words, at any fixed point ξ on the buying boundary β_i,

$$E\left[dL_i \mid \xi\right] = -\frac{u(\xi)}{2\nu_i} \nu_j a_{jk} a_{lk} \nu_l dt,$$

where ν is the unit vector normal to β_i at ξ that points out of the hold region. By the same logic, at any fixed point ξ on the selling surface γ_i,

$$E\left[dM_i \mid \xi\right] = \frac{u(\xi)}{2\nu_i} \nu_j a_{jk} a_{lk} \nu_l dt,$$

where, again, ν is the unit vector normal to γ_i at ξ that points out of the hold region.

Given these results, the differential loss of expected utility (84) takes the form

$$\begin{aligned} E[df] &= \frac{f_{zz}}{2} \left(\int_{\mathcal{H}} \xi_i \sigma_{ik} \sigma_{jk} \xi_j u(\xi) d\xi \right) dt \\ &\quad + \varepsilon f_z A_{jl} \left(b_i \int_{\beta_i} \frac{\nu_j(\xi)\nu_l(\xi)}{\nu_i(\xi)} u(\xi) dS - c_i \int_{\gamma_i} \frac{\nu_j(\xi)\nu_l(\xi)}{\nu_i(\xi)} u(\xi) dS \right) dt \end{aligned} \tag{88}$$

where

$$A_{ij} = \frac{1}{2} a_{ik} a_{jk} \tag{89}$$

is a symmetric, positive semi-definite matrix. Also note that ν_i is negative on any β_i and positive on any γ_i so, since $f_{zz} < 0$ and $f_z > 0$, we have that both the opportunity cost and the trading cost are negative quantities.

Now we look to determine constraints on u that come from the forward Kolmogorov equation. We derive this equation by starting with the Hamilton-Jacobi-Bellman equation for f corresponding to the dynamics in (85), which is

$$f_t = -A_{ij} f_{\xi_i \xi_j} \qquad \xi \in \mathcal{H} \tag{90}$$

subject to the Neumann boundary condition

$$f_{\xi_k} = 0 \qquad \xi \in \beta_k \cup \gamma_k, \tag{91}$$

where $k = 1, 2, ..., n$. From the steady state of the dynamics, we have that

$$0 = \frac{d}{dt} \int_{\mathcal{H}} u f d\xi, \tag{92}$$

and we can now use the duality of u and f to find the corresponding forward Kolmogorov equation for u. Specifically, from applying (90), integration by parts, and the symmetry of A_{ij} to (92) we have

$$\begin{aligned} 0 &= \int_{\mathcal{H}} u_t f + u f_t d\xi \\ &= \int_{\mathcal{H}} u_t f - u A_{ij} f_{\xi_i \xi_j} d\xi \\ &= \int_{\mathcal{H}} \left(u_t - A_{ij} u_{\xi_i \xi_j} \right) f d\xi + A_{ij} \int_{\partial \mathcal{H}} u_{\xi_i} \nu_j f - u \nu_j f_{\xi_i} dS. \end{aligned}$$

From the integral over $\mathcal{H}$ in the last line and the steady state dynamics we obtain the PDE constraint on u

$$0 = A_{ij} u_{\xi_i \xi_j} \qquad \xi \in \mathcal{H}, \tag{93}$$

and from the integral over $\partial \mathcal{H}$ in the last line we have

$$0 = A_{ij} \int_{\partial \mathcal{H}} u_{\xi_i} \nu_j f - u \nu_j f_{\xi_i} dS. \tag{94}$$

We would like to use integration by parts on the f_{ξ_i} in (94) so that we subsequently factor f and conclude that the remainder of the integrand is zero as we just did with the integral over $\mathcal{H}$, but we cannot apply integration by parts with the partial derivative operator on the $\partial \mathcal{H}$ manifold.

We now make a number of definitions that will lead to a differential operator for which the necessary integration by parts can be applied to the integral over the boundary, $\partial\mathcal{H}$. First, we define $\hat{\xi}_k$ to be the $(n-1)$-dimensional space $(\xi_1, \xi_2, ..., \xi_{k-1}, \xi_{k+1}, ..., \xi_n)$. Next we define B_k (and, respectively, Γ_k) to be the orthogonal projection of β_k (respectively, γ_k) into $\hat{\xi}_k$ space, and now we can define the function $\hat{\beta}_k$ over the domain B_k (and, respectively, $\hat{\gamma}_k$ over the domain Γ_k) so that the graph of the function $\hat{\beta}_k$ is the set β_k (and the graph of $\hat{\gamma}_k$ is γ_k). Finally, we define the *function* $\beta_k(\cdot)$ — as opposed to the previously defined *set* β_k — over the domain $\left(\hat{\xi}_k, \xi_k\right) \in (B_k, \mathbf{R})$ by

$$\beta_k(\xi) = \hat{\beta}_k\left(\hat{\xi}_k\right) - \xi_k, \tag{95}$$

so the level set $\beta_k(\xi) = 0$ of the *function* is the *set* β_k where we buy stock k. Similarly, we define the function γ_k so that the level set $\gamma_k(\xi) = 0$ is the set γ_k where we sell stock k. Note: throughout the remainder of this paper, when we are using β_k or γ_k as functions, not sets, we will either explicitly write the functional dependence (i.e., $\beta_k(\xi)$ and $\gamma_k(\xi)$) or, as will be more common, it will be clear because we are taking derivatives (e.g., $(\beta_k)_{\xi_i}$ only makes sense for the *function* β_k).

Note that for any generic function $g(\xi)$ restricted to the set β_k (and the same logic will apply to being restricted to γ_i), we have $g\left(\hat{\xi}_k, \hat{\beta}_k\left(\hat{\xi}_k\right)\right)$ where $\hat{\xi}_k \in B_k$. We will use the notation d_{ξ_i} to denote "total" differentiation in B_k space, so in particular, by the chain rule, we have that $d_{\xi_i} g = g_{\xi_i} + g_{\xi_k}\left(\hat{\beta}_k\right)_{\xi_i}$, where $i \neq k$ and we do not sum over k. Further, since $d_{\xi_k} g = 0$ on β_k and $(\beta_k)_{\xi_k} = -1$, we can extend this chain rule to include the $i = k$ case:

$$d_{\xi_i} g = g_{\xi_i} + g_{\xi_k} (\beta_k)_{\xi_i} \qquad i = 1, 2, ..., n\ , \tag{96}$$

where, again, we do not sum over k. The key point is that we can apply integration by parts to the d_{ξ_i} operator when we express the boundary integration over the B_k (and Γ_k) domains. In light of this and the geometric fact that

$$\nu_i dS = (\beta_k)_{\xi_i} d\hat{\xi}_k \tag{97}$$

and

$$\nu_i dS = -(\gamma_k)_{\xi_i} d\hat{\xi}_k \tag{98}$$

—where, again, we do not sum over k—we now rewrite the boundary integral equation (94) as

$$0 = A_{ij}\left(\int_{B_k} u_{\xi_i} (\beta_k)_{\xi_j} f - u (\beta_k)_{\xi_j} f_{\xi_i} d\hat{\xi}_k - \int_{\Gamma_k} u_{\xi_i} (\gamma_k)_{\xi_j} f - u (\gamma_k)_{\xi_j} f_{\xi_i} d\hat{\xi}_k\right). \tag{99}$$

However, the chain rule (96) and the Neumann boundary condition (91) imply that $d_{\xi_i} f = f_{\xi_i}$, so we can substitute $d_{\xi_i} f$ for f_{ξ_i} in (99) and perform the desired integration by parts to yield

$$0 = A_{ij} \left(\begin{array}{c} \int_{B_k} \left[u_{\xi_i} \left(\beta_k\right)_{\xi_j} + d_{\xi_i} \left(u \left(\beta_k\right)_{\xi_j} \right) \right] f d\hat{\xi}_k \\ - \int_{\Gamma_k} \left[u_{\xi_i} \left(\gamma_k\right)_{\xi_j} + d_{\xi_i} \left(u \left(\gamma_k\right)_{\xi_j} \right) \right] f d\hat{\xi}_k \end{array} \right) + I^{n-2},$$

where I^{n-2} represents the $(n-2)$-dimensional integrals produced by the integration by parts on the boundaries of the sets B_k and Γ_k. With f now factored in the $(n-1)$-dimensional integrals over B_k and Γ_k, we have our desired oblique derivative boundary conditions on u:

$$\begin{array}{rcll} 0 & = & A_{ij} \left[u_{\xi_i} \left(\beta_k\right)_{\xi_j} + d_{\xi_i} \left(u \left(\beta_k\right)_{\xi_j} \right) \right] & \text{on } B_k \\ 0 & = & -A_{ij} \left[u_{\xi_i} \left(\gamma_k\right)_{\xi_j} + d_{\xi_i} \left(u \left(\gamma_k\right)_{\xi_j} \right) \right] & \text{on } \Gamma_k. \end{array} \tag{100}$$

We can also use that $I^{n-2} = 0$ to obtain restrictions on the ways in which the $2n$ buying and selling boundaries may intersect, but these conditions are unnecessary for our purposes here so we will not pursue them further. Of course, there is a final constraint on u since the integral of the probability over the hold region must equal 1; that is,

$$\int_{\mathcal{H}} u(\xi) d\xi = 1. \tag{101}$$

We now have (93), (100), and (101), which form the three probability constraints that we will need in Section 3.4 to more rigorously justify (88), our expression showing the loss in expected utility is the sum of the opportunity and trading costs.

3.3 Asymptotic Analysis for Multiple Stocks

As in Section 2.3, we now consider our problem from the point of view of a more rigorous asymptotic approximation for the value function, f. We expect the diameter of the hold region to be $O\left(\varepsilon^{\frac{1}{3}}\right)$ just as it was in the single stock case, and so, as we did in the single stock case, we will look for an expansion of f in powers of $\varepsilon^{\frac{1}{3}}$ and we will rescale ξ so that ξ from the previous section is now $\varepsilon^{\frac{1}{3}}\xi$ in this section, and, as in Section 2, we will revert back Section 3.2's scaling in Section 3.4. Therefore, the differentials dZ in

(81) and $d\xi_i$ in (82) now take the form

$$dZ = \left[(\mu_i - r)\,(m_i + \varepsilon^{\frac{1}{3}}\xi_i) + rz\right] dt + \left[(m_i + \varepsilon^{\frac{1}{3}}\xi_i)\sigma_{ij}\right] dB_j - \varepsilon\,(b_i dL_i + c_i dM_i) \tag{102}$$

and

$$\begin{aligned} d\xi_i &= dX_i - (m_i)_t\,dt - (m_i)_z\,dZ - \frac{1}{2}(m_i)_{zz}(dZ)^2 \\ &= \frac{1}{\varepsilon^{\frac{1}{3}}}\left\{ \begin{array}{c} \left[\begin{array}{c} \mu_i\left(m_i + \varepsilon^{\frac{1}{3}}\xi_i\right) - (m_i)_t \\ -(m_i)_z\left((\mu_j - r)\left(m_j + \varepsilon^{\frac{1}{3}}\xi_j\right) + rz\right) \\ -\frac{1}{2}(m_i)_{zz}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\sigma_{jk}\sigma_{lk}(m_l + \varepsilon^{\frac{1}{3}}\xi_l) \end{array} \right] dt \\ + \left[a_{ik} + \varepsilon^{\frac{1}{3}}\alpha_{ik}\right] dB_k \\ + (1 - \varepsilon b_i)\,dL_i + \varepsilon\,(m_i)_z\,b_j dL_j - dM_i + \varepsilon\,(m_i)_z\,a_j dM_j \end{array} \right\} \end{aligned} \tag{103}$$

where the abbreviation a_{ij} was defined in (86) and the abbreviation α_{ij} is defined by:

$$\alpha_{ij} = -\frac{\partial m_i}{\partial z}\xi_k\sigma_{kj} + \xi_i\sigma_{ij}.$$

In the interior of the hold region where $dL_i = dM_i = 0$, these differentials lead to the following Hamilton-Jacobi-Bellman equation for $f(z,\xi,t) = \max\left\{E_{z,\xi,t}\left[U\left(z(T)\right)\right]\right\}$:

$$\begin{aligned} 0 &= f_t + \left[(\mu_j - r)\left(m_j + \varepsilon^{\frac{1}{3}}\xi_j\right) + rz\right] f_z \\ &\quad + \frac{1}{\varepsilon^{\frac{1}{3}}}\left[\begin{array}{c} \mu_i\left(m_i + \varepsilon^{\frac{1}{3}}\xi_i\right) - (m_i)_t - (m_i)_z\left((\mu_j - r)\left(m_j + \varepsilon^{\frac{1}{3}}\xi_j\right) + rz\right) \\ -\frac{1}{2}(m_i)_{zz}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\sigma_{jk}\sigma_{lk}(m_l + \varepsilon^{\frac{1}{3}}\xi_l) \end{array} \right] f_{\xi_i} \\ &\quad + \frac{1}{2}\frac{1}{\varepsilon^{\frac{2}{3}}}\left[\left(a_{ik} + \varepsilon^{\frac{1}{3}}\alpha_{ik}\right)\left(a_{jk} + \varepsilon^{\frac{1}{3}}\alpha_{jk}\right)\right] f_{\xi_i\xi_j} \\ &\quad + \frac{1}{\varepsilon^{\frac{1}{3}}}\left[\left(a_{ik} + \varepsilon^{\frac{1}{3}}\alpha_{ik}\right)\sigma_{jk}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\right] f_{\xi_i z} \\ &\quad + \frac{1}{2}\left[(m_i + \varepsilon^{\frac{1}{3}}\xi_i)\sigma_{ik}\sigma_{jk}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\right] f_{zz} \end{aligned} \tag{104}$$

On the boundary of the hold region, dL_k dominates the Hamilton-Jacobi-Bellman equation when we are on β_k, and dM_k dominates when we are on γ_k, which implies the first derivative boundary conditions

$$0 = -\varepsilon b_k f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k} - \varepsilon^{\frac{2}{3}} b_k f_{\xi_k} - \varepsilon^{\frac{2}{3}} b_k\,(m_j)_z\,f_{\xi_j} \qquad \text{on } \beta_k \tag{105}$$

$$0 = \varepsilon a_k f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k} - \varepsilon^{\frac{2}{3}} c_k\,(m_j)_z\,f_{\xi_j} \qquad \text{on } \gamma_k. \tag{106}$$

(We do not sum over k here.) From Appendix 1 we know that on the boundary of the hold region that optimizes $E[df]$ we have a second derivative boundary condition that can be obtained by differentiating the first derivative boundary conditions in any direction transverse to the boundary. By choosing this direction to be ξ_k on β_k or γ_k, we get from (105) and (106) that the second derivative optimization condition is

$$
\begin{aligned}
0 &= -\varepsilon b_k f_{z\xi_k} + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k\xi_k} - \varepsilon^{\frac{2}{3}} b_k f_{\xi_k\xi_k} - \varepsilon^{\frac{2}{3}} b_k \left(m_j\right)_z f_{\xi_j\xi_k} \qquad \text{on } \beta_k && (107) \\
0 &= \varepsilon c_k f_{z\xi_k} + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k\xi_k} - \varepsilon^{\frac{2}{3}} c_k \left(m_j\right)_z f_{\xi_j\xi_k} \qquad \text{on } \gamma_k. && (108)
\end{aligned}
$$

(Again, we do not sum over k here.)

Looking at the first order boundary conditions, (105) and (106), we see that the $\frac{\partial f}{\partial z}$ term is $O\left(\varepsilon^{\frac{4}{3}}\right)$ larger than the highest order term for $\frac{\partial f}{\partial \xi_k}$. This implies that ξ affects the solution for f only at the $O\left(\varepsilon^{\frac{4}{3}}\right)$ level, and so we consider an expansion for f of the form

$$
f = f^o(z,t) + \varepsilon^{\frac{1}{3}} f^1(z,t) + \varepsilon^{\frac{2}{3}} f^2(z,t) + \varepsilon f^3(z,t) + \varepsilon^{\frac{4}{3}} f^4(z,\xi,t) + \ldots \qquad (109)
$$

Given this expansion, the highest order terms in both the first and second derivative boundary conditions are $O\left(\varepsilon\right)$, which, specifically, from (105) and (106) are

$$
\begin{aligned}
\left(f^4\right)_{\xi_k} &= b_k \left(f^0\right)_z \qquad \text{on } \beta_k \\
\left(f^4\right)_{\xi_k} &= -c_k \left(f^0\right)_z \qquad \text{on } \gamma_k && (110)
\end{aligned}
$$

and from (107) and (108) are

$$
\left(f^4\right)_{\xi_k\xi_k} = 0 \qquad \text{on } \beta_k \text{ or } \gamma_k. \qquad (111)
$$

We begin to set up the asymptotic expansion by first collecting terms in (104) with identical powers of $\varepsilon^{\frac{1}{3}}$:

$$
\begin{aligned}
0 = {} & \frac{1}{\varepsilon^{\frac{2}{3}}} \left[\begin{array}{c} \frac{1}{2}\left(1-\left(m_i\right)_z\right) m_l \sigma_{lk} \\ \cdot \sigma_{mk} \left(1-\left(m_j\right)_z\right) m_m f_{\xi_i\xi_j} \end{array} \right] \\
& + \frac{1}{\varepsilon^{\frac{1}{3}}} \left[\begin{array}{c} \left(\mu_i m_i - \left(m_i\right)_t - \left(m_i\right)_z \left(\left(\mu_j - r\right) m_j + rz\right)\right. \\ \left. -\frac{1}{2}\left(m_i\right)_{zz} m_j \sigma_{ij} \sigma_{kj} m_k\right) f_{\xi_i} \\ + a_{ik} a_{jk} f_{\xi_i\xi_j} + a_{ik} \sigma_{jk} m_j f_{z\xi_i} \end{array} \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon^0 \left[\begin{array}{c} f_t + \left(\left(\mu_i - r \right) m_i + rz \right) f_z \\ + \left(\mu_i \xi_i - \left(m_i \right)_z \left(\left(\mu_j - r \right) \xi_j \right) - \left(m_i \right)_{zz} m_j \sigma_{jk} \sigma_{lk} \xi_l \right) f_{\xi_i} \\ + \frac{1}{2} \alpha_{ik} \alpha_{jk} f_{\xi_i \xi_j} \\ + \left(a_{ik} \sigma_{jk} \xi_j + \alpha_{ik} \sigma_{jk} m_j \right) f_{z \xi_i} + \frac{1}{2} m_i \sigma_{ik} \sigma_{jk} m_j f_{zz} \end{array} \right] \\
&+\varepsilon^{\frac{1}{3}} \left[\begin{array}{c} \left(\mu_j - r \right) \xi_j f_z - \frac{1}{2} \left(m_i \right)_{zz} \xi_j \sigma_{jk} \sigma_{lk} \xi_l f_{\xi_i} \\ + \alpha_{ik} \sigma_{jk} \xi_j f_{z \xi_i} + m_i \sigma_{ik} \sigma_{jk} \xi_j f_{zz} \end{array} \right] \\
&+\varepsilon^{\frac{2}{3}} \left[\frac{1}{2} \xi_i \sigma_{ik} \sigma_{jk} \xi_j f_{zz} \right] \qquad (112)
\end{aligned}
$$

and then, after inserting (109), the expansion for f, into (112), we again collect terms with identical powers of $\varepsilon^{\frac{1}{3}}$, each of which we set to zero. The order ε^0 terms from this expansion lead, of course, to the Merton (no transaction costs) equation for f^0:

$$
0 = \left(f^0 \right)_t + \left(\left(\mu_i - r \right) m_i + rz \right) \left(f^0 \right)_z + \frac{1}{2} m_i \sigma_{ik} \sigma_{jk} m_j \left(f^0 \right)_{zz} .
$$

The order $\varepsilon^{\frac{1}{3}}$ terms contain both f^0 and f^1 parts:

$$
\begin{aligned}
0 &= \xi_j \left[\left(\mu_j - r \right) \left(f^0 \right)_z + m_i \sigma_{ik} \sigma_{jk} \left(f^0 \right)_{zz} \right] \\
&\quad + \left[\left(f^1 \right)_t + \left(\left(\mu_i - r \right) m_i + rz \right) \left(f^1 \right)_z + \frac{1}{2} m_i \sigma_{ik} \sigma_{jk} m_j \left(f^1 \right)_{zz} \right] . \qquad (113)
\end{aligned}
$$

Since, from (77), the Merton values are

$$
m_i = - \frac{\left(\sigma_{ik} \sigma_{jk} \right)^{-1} \left(\mu_j - r \right) \left(f^0 \right)_z}{\left(f^0 \right)_{zz}} ,
$$

we see, by substitution, that the f^0 terms in (113) cancel each other, which leaves the Merton equation for f^1, but since the condition at T for f^1 is $f^1(z,T) = 0$, we must have, by uniqueness, that $f^1(z,t) = 0$. Knowing that $f^1 = 0$, we now collect the remaining order $\varepsilon^{\frac{2}{3}}$ terms:

$$
\begin{aligned}
0 &= A_{ik} \left(f^4 \right)_{\xi_i \xi_j} + \frac{1}{2} \xi_i \sigma_{ik} \sigma_{jk} \xi_j \left(f^0 \right)_{zz} \\
&\quad + \left(f^2 \right)_t + \left(\left(\mu_i - r \right) m_i + rz \right) \left(f^2 \right)_z + \frac{1}{2} m_i \sigma_{ik} \sigma_{jk} m_j \left(f^2 \right)_{zz} , \qquad (114)
\end{aligned}
$$

where the symmetric, positive semi-definite matrix A_{ij} was defined in (89). As in the single stock case, we assume A_{ij} is positive definite, so we have an elliptic PDE in the ξ variables for f^4 of the form

$$
K = A_{ij} \left(f^4 \right)_{\xi_i \xi_j} + \frac{1}{2} \left(f^0 \right)_{zz} \xi_i \sigma_{ik} \sigma_{jk} \xi_j \qquad (115)
$$

where $-K$ is comprised of all three terms in the second line of (114), and we note that A_{ij}, K, σ, and $(f^0)_{zz}$ have no dependence on ξ.

3.4 Lagrange Multiplier Perspective

As in the single stock case, we now look to find a deeper connection between the less rigorous, but more intuitive, analysis of Section 3.2 and the asymptotic expansions of Section 3.3. Again, we will find that the Lagrange multiplier function that is dual to the probability constraints in the constrained optimization described in Section 3.2 corresponds to f_4, the leading order term depending on ξ in the maximum expected utility expansion from Section 3.3.

We begin with the constrained optimization problem of Section 3.2. We are interested in optimizing $E[df]$, which is given by (88), subject to the PDE constraint (93) within the body of the hold region, the Neumann constraint (100) on the boundary of the hold region, and the integrability constraint for probability (101). Recall that in Section 3.2, the Neumann constraint (100) was derived and expressed in the projected spaces B_k and Γ_k so that integration by parts with the "total" derivative operator d_{ξ_i} was possible. We will need to use these same techniques here, so we must express the boundary integral in our expression for $E[df]$ as integrals over the projected spaces B_k and Γ_k. This is accomplished by applying the geometric facts (97), (98) , and

$$\begin{aligned} \frac{\nu_i}{\nu_k} &= -(\beta_k)_{\xi_i} && \text{on } \beta_k, \\ \frac{\nu_i}{\nu_k} &= -(\gamma_k)_{\xi_i} && \text{on } \gamma_k \end{aligned}$$

to the boundary integral in (88), which yields the form of function we want to optimize

$$\begin{aligned} \frac{E[df]}{dt} &= \frac{f_{zz}}{2}\left(\int_{\mathcal{H}} \xi_i \sigma_{ik}\sigma_{jk}\xi_j u(\xi) d\xi\right) \\ &\quad -\varepsilon f_z A_{ij}\left(\begin{array}{c} b_k \int_{B_k} (\beta_k)_{\xi_i} (\beta_k)_{\xi_j} u(\xi) d\hat{\xi}_k \\ +c_k \int_{\Gamma_k} (\gamma_k)_{\xi_i} (\gamma_k)_{\xi_j} u(\xi) d\hat{\xi}_k \end{array}\right). \end{aligned} \tag{116}$$

As in the single variable case, we use λ to denote the Lagrange multiplier function defined in the hold region that corresponds to the PDE constraint (93), μ to denote the Lagrange multiplier function defined on the boundary of the hold region that corresponds to the Neumann constraint (100), and ν to denote the scalar Lagrange multiplier that corresponds to the integrability constraint (101) (and not to be confused with ν, the outward normal to the boundary, which we have now removed from our expressions). Given this, the expression (146) in Appendix 2 takes the form

$$
\begin{aligned}
&\frac{f_{zz}}{2}\left(\int_{\mathcal{H}} \xi_i \sigma_{ik} \sigma_{jk} \xi_j u d\xi\right) \\
&-\varepsilon f_z A_{ij}\left(b_k \int_{B_k} (\beta_k)_{\xi_i} (\beta_k)_{\xi_j} u d\hat{\xi}_k + c_k \int_{\Gamma_k} (\gamma_k)_{\xi_i} (\gamma_k)_{\xi_j} u d\hat{\xi}_k\right) \\
&-A_{ij} \int_{\mathcal{H}} \lambda u_{\xi_i \xi_j} d\xi \\
&-A_{ij} \int_{B_k} \mu \left[u_{\xi_i} (\beta_k)_{\xi_j} + d_{\xi_i}\left(u (\beta_k)_{\xi_j}\right)\right] d\hat{\xi}_k \\
&+A_{ij} \int_{\Gamma_k} \mu \left[u_{\xi_i} (\gamma_k)_{\xi_j} + d_{\xi_i}\left(u (\gamma_k)_{\xi_j}\right)\right] d\hat{\xi}_k \\
&-\nu\left(\int_{\mathcal{H}} u d\xi - 1\right).
\end{aligned}
$$

(Although this is not an equation, we sum over all indices.) We apply integration by parts twice to the $\int_{\mathcal{H}} \lambda u_{\xi_i \xi_j} d\xi$ integral and then apply the chain rule (96) and integration by parts with the d_{ξ_i} operator on the resulting B_k and Γ_k integrals, which, after some algebra, yields

$$
\begin{aligned}
&\int_{\mathcal{H}} \left(-A_{ij} \lambda_{\xi_i \xi_j} + \frac{f_{zz}}{2} \xi_i \sigma_{ik} \sigma_{jk} \xi_j - \nu\right) u d\xi \\
&+A_{ij} \int_{B_k} \left[\begin{array}{c} -\varepsilon f_z b_k (\beta_k)_{\xi_i} (\beta_k)_{\xi_j} + \lambda_{\xi_i} (\beta_k)_{\xi_j} \\ +d_{\xi_i}\left((2\mu + \lambda) (\beta_k)_{\xi_j}\right) - \mu d_{\xi_i}\left((\beta_k)_{\xi_j}\right) \end{array}\right] u \\
&+\left[(\mu + \lambda) (\beta_k)_{\xi_i} (\beta_k)_{\xi_j}\right] u_{\xi_k} d\hat{\xi}_k \\
&-A_{ij} \int_{\Gamma_k} \left[\begin{array}{c} \varepsilon f_z c_k (\gamma_k)_{\xi_i} (\gamma_k)_{\xi_j} + \lambda_{\xi_i} (\gamma_k)_{\xi_j} \\ +d_{\xi_i}\left((2\mu + \lambda) (\gamma_k)_{\xi_j}\right) - \mu d_{\xi_i}\left((\gamma_k)_{\xi_j}\right) \end{array}\right] u \\
&+\left[(\mu + \lambda) (\gamma_k)_{\xi_i} (\gamma_k)_{\xi_j}\right] u_{\xi_k} d\hat{\xi}_k \\
&+\nu + I^{n-2}, \qquad (117)
\end{aligned}
$$

where, again, we sum over all indices, and, as in Section 3.2, I^{n-2} represents the $(n-2)$-dimensional integrals produced from the integration by parts on the boundaries of the sets B_k and Γ_k. Note that in this form u and u_{ξ_k} have been factored out, which makes it easy to perturb u in (117) to obtain the Lagrange multiplier equation (147) from Appendix 2:

$$\begin{aligned}
0 &= \int_{\mathcal{H}} \left(-A_{ij}\lambda_{\xi_i\xi_j} + \frac{f_{zz}}{2}\xi_i\sigma_{ik}\sigma_{jk}\xi_j - \nu \right) \dot{u} d\xi \\
&\quad + A_{ij} \int_{B_k} \left[\begin{array}{c} -\varepsilon f_z b_k \left(\beta_k\right)_{\xi_i} \left(\beta_k\right)_{\xi_j} + \lambda_{\xi_i} \left(\beta_k\right)_{\xi_j} \\ +d_{\xi_i} \left((2\mu + \lambda) \left(\beta_k\right)_{\xi_j} \right) - \mu d_{\xi_i} \left(\left(\beta_k\right)_{\xi_j} \right) \end{array} \right] \dot{u} \\
&\quad + \left[(\mu + \lambda) \left(\beta_k\right)_{\xi_i} \left(\beta_k\right)_{\xi_j} \right] \dot{u}_{\xi_k} d\hat{\xi}_k \\
&\quad - A_{ij} \int_{\Gamma_k} \left[\begin{array}{c} \varepsilon f_z c_k \left(\gamma_k\right)_{\xi_i} \left(\gamma_k\right)_{\xi_j} + \lambda_{\xi_i} \left(\gamma_k\right)_{\xi_j} \\ +d_{\xi_i} \left((2\mu + \lambda) \left(\gamma_k\right)_{\xi_j} \right) - \mu d_{\xi_i} \left(\left(\gamma_k\right)_{\xi_j} \right) \end{array} \right] \dot{u} \\
&\quad + \left[(\mu + \lambda) \left(\gamma_k\right)_{\xi_i} \left(\gamma_k\right)_{\xi_j} \right] \dot{u}_{\xi_k} d\hat{\xi}_k. \qquad (118)
\end{aligned}$$

Here, we do not consider perturbing the probability at the (assumed optimal) intersections of the $2n$ buying and selling boundaries, thereby removing the effect of I^{n-2}.

Since $\dot{u}$ is otherwise arbitrary, we get from the first line of (118), the following PDE within $\mathcal{H}$:

$$0 = -A_{ij}\lambda_{\xi_i\xi_j} + \frac{f_{zz}}{2}\xi_i\sigma_{ik}\sigma_{jk}\xi_j - \nu. \qquad (119)$$

Since the matrix A_{ij} is positive definite, the third and fifth lines of (118) reduce to

$$\mu + \lambda = 0 \qquad (120)$$

on the boundary of $\mathcal{H}$. Applying (120) to the second and fourth lines of (118) along with the product rule, the chain rule (96), and A_{ij} being positive definite yields Neumann conditions for λ :

$$\begin{aligned}
\lambda_{\xi_k} &= -\varepsilon b_k f_z \qquad \text{on } \beta_k \\
\lambda_{\xi_k} &= \varepsilon c_k f_z \qquad \text{on } \gamma_k. \qquad (121)
\end{aligned}$$

Next we look at the effect of perturbing the boundary in (117) to form the optimality equation (148) from Appendix 2. Infinitesimally perturbing the boundary function $\hat{\beta}_k\left(\hat{\xi}_k\right)$ causes a corresponding perturbation in $\beta_k(\xi)$ (with no ξ_k dependence), which we will call $\dot{\beta}_k$. (We will perturb γ_k later.) As in the single stock case, equations (119), (120), and (121), which resulted from the Lagrange multiplier equation, imply that the effect of perturbing the boundary on $\mathcal{H}$, $\partial\mathcal{H}$, u, and u_{ξ_k} in (117) is zero. Therefore, nonzero terms in the optimality equation result solely from the effect of the boundary perturbation on λ, the derivatives of λ, and the derivatives of $\beta(\xi)$ in the boundary integrals of (117). Since $\dot{\beta}_k = \dot{\hat{\beta}}_k$, we have by the chain rule that, for example, $\dot{\lambda}\left(\hat{\xi}_k, \hat{\beta}_k\left(\hat{\xi}_k\right)\right) = \lambda_{\xi_k} \dot{\hat{\beta}}_k = \lambda_{\xi_k}\dot{\beta}_k$, and so, applying the symmetry of A_{ij}, the optimality equation (148) from Appendix 2 takes the form

$$
\begin{aligned}
0 \quad = \quad & A_{ij}\int_{B_k}\left[\begin{array}{c} -\varepsilon f_z b_k \left(2\left(\beta_k\right)_{\xi_i}\left(\dot{\beta}_k\right)_{\xi_j}\right) + \lambda_{\xi_i\xi_k}\dot{\beta}_k\left(\beta_k\right)_{\xi_j} + \lambda_{\xi_i}\left(\dot{\beta}_k\right)_{\xi_j} \\ +d_{\xi_i}\left[\lambda_{\xi_k}\dot{\beta}_k\left(\beta_k\right)_{\xi_j} + (2\mu+\lambda)\left(\dot{\beta}_k\right)_{\xi_j}\right] - \mu d_{\xi_i}\left(\left(\dot{\beta}_k\right)_{\xi_j}\right) \end{array}\right] u \\
& + \left[\lambda_{\xi_k}\dot{\beta}_k\left(\beta_k\right)_{\xi_i}\left(\beta_k\right)_{\xi_j} + (\mu+\lambda)\left(2\left(\beta_k\right)_{\xi_i}\left(\dot{\beta}_k\right)_{\xi_j}\right)\right] u_{\xi_k} d\hat{\xi}_k. \qquad (122)
\end{aligned}
$$

(where, again, we do not perturb the boundary at the assumed optimal intersections of the buying and selling boundaries). Since $\dot{\beta}_k$ has no ξ_k dependence, we have from the chain rule (96) that $d_{\xi_i}\dot{\beta}_k = \left(\dot{\beta}_k\right)_{\xi_i}$. Applying this fact, (120), (121), the chain rule (96), the product rule, integration by parts with the d_{ξ_i} operator, and a significant amount of algebra allows us to reduce (122) to

$$
\begin{aligned}
0 \quad = \quad & A_{ij}\int_{B_k}\varepsilon f_z b_k\left[u_{\xi_i}\left(\beta_k\right)_{\xi_j} + d_{\xi_i}\left(u\left(\beta_k\right)_{\xi_j}\right)\right]\dot{\beta}_k \\
& + \left(d_{\xi_i}\left(\lambda_{\xi_k}\right) - \lambda_{\xi_k\xi_k}\left(\beta_k\right)_{\xi_i}\right)\left(\beta_k\right)_{\xi_j}\dot{\beta}_k u d\hat{\xi}_k. \qquad (123)
\end{aligned}
$$

By the Neumann condition (100), the first line of (123) is zero, and by (121), $d_{\xi_i}\left(\lambda_{\xi_k}\right)$ in the second line is zero. Since the perturbation $\dot{\beta}_k$ is arbitrary, we have that $0 = A_{ij}\left(\beta_k\right)_{\xi_i}\left(\beta_k\right)_{\xi_j} u\lambda_{\xi_k\xi_k}$ on β_k. By the same logic, an arbitrary perturbation of γ_k leads to $0 = A_{ij}\left(\gamma_k\right)_{\xi_i}\left(\gamma_k\right)_{\xi_j} u\lambda_{\xi_k\xi_k}$ on γ_k. Since $u > 0$ and A_{ij} is positive definite, the final form of the optimality equation is the second derivative boundary condition

$$
\lambda_{\xi_k\xi_k} = 0 \qquad \text{on } \beta_k \text{ or } \gamma_k. \qquad (124)
$$

If we identify $\lambda(\xi)$ with

$$
-\varepsilon^{\frac{4}{3}} f_4\left(z, \varepsilon^{-\frac{1}{3}}\xi, t\right),
$$

we see that our PDE, (119), first derivative boundary conditions, (121), and second derivative boundary conditions, (124), completely match Section 3.3's PDE, (115), first derivative boundary conditions, (110), and second derivative boundary conditions, (111). Therefore, the asymptotic analysis from Section 3.3 justifies the assumptions in Section 3.2 that were made to determine (88), our characterization of the optimization problem in terms of the opportunity and trading costs.

3.5 Classical Solution in the Uncorrelated Multiple Stock Case

In this section, we work with the results from Section 3.3 so we again use that section's scaling: specifically, ξ in Section 3.2 and 3.4 is again replaced by $\varepsilon^{\frac{1}{3}}\xi$ as in Section 3.3. Generally, it is difficult to find a hold region and a function, f^4, that satisfies the elliptic PDE, (115), and the boundary conditions, (110) and (111). We note as an exception the case where $\sigma_{ij} = \sigma_i \delta_{ij}$; that is, the case where the n stocks are uncorrelated so the matrix σ is diagonal with elements σ_i on the diagonal. In this case, we have the following generalization of the single stock results: The hold region in ξ space is an n-dimensional box centered at the origin where $\tilde{\gamma}_i$, the half-width in the i direction, is given by

$$\tilde{\gamma}_i = \left(\frac{3}{2} A_{ii} \frac{(b_i + c_i) f_z^0}{\sigma_i^2 f_{zz}^0} \right)^{\frac{1}{3}} .$$

In other words, $-\hat{\beta}_i \left(\hat{\xi}_i\right) = \hat{\gamma}_i \left(\hat{\xi}_i\right) = \tilde{\gamma}_i$. In this case, the function f^4 takes the form

$$f^4(z, \xi, t) = C_i^1(z,t) \left(\xi_i\right)^4 + C_i^2(z,t) \left(\xi_i\right)^2 + C_i^3(z,t)\xi_i + C^4(z,t) \qquad (125)$$

where

$$\begin{aligned} C_i^1(z,t) &= -\frac{(b_i + c_i) f_z^0}{16 \left(\tilde{\gamma}_i\right)^3}, \\ C_i^2(z,t) &= -6 C_i^1(z,t) \left(\tilde{\gamma}_i\right)^2, \\ C_i^3(z,t) &= \frac{(b_i - c_i) f_z^0}{2}, \end{aligned}$$

and the "constant" function $C^4(z,t)$ is indeterminable. Note that this form for f^4 will determine K, and therefore f^2.

It is clear from (125) that in the interior of the hold region $\mathcal{H}$, we have that f^4 is a smooth function of the microscopic variables, ξ_i; specifically, it is a quartic polynomial with no cross-terms. Since the macroscopic variables z and t essentially behave like constants near the hold region, we see that, to leading order, the solution is classical if it is also smooth in ξ_i at $\partial\mathcal{H}$, the boundary of the hold region. We will show in this section that this is the case. Specifically, we will show that the solution in the hold region and the solution in the trading region (that is, outside of the hold region) are C^2 matching at the boundary, $\partial\mathcal{H}$, between them.

We begin, for the sake of intuition, by analyzing the trading region in the single stock case. In this case, the trading region can be split into the buy region and the sell region. If we are in either region, the opportunity cost outweighs the trading cost, so we immediately buy or sell stock to move the portfolio to the boundary of the hold region. Consider the case of the buy region, $\xi \leq -\tilde{\gamma}$. For any $\xi_1 < \xi_2 \leq -\tilde{\gamma}$, the fact that we immediately buy stock implies from the nature of the trading costs and the definition of ξ that

$$f(z, \xi_1, t) = f\left(z - \frac{\varepsilon b}{\varepsilon^{-\frac{1}{3}} - (1 - m_z)\varepsilon^{\frac{2}{3}}}(\xi_2 - \xi_1), \xi_2, t\right). \tag{126}$$

Letting ξ_2 approach ξ_1, we obtain, to leading order in ε, the differential equation

$$-\varepsilon^{\frac{4}{3}} b f_z + f_\xi = 0,$$

and, after inserting the expansion for f into this equation, it becomes, to leading order in ε,

$$-b f_z^0(z, t) + f_\xi^4(z, \xi, t) = 0.$$

Since z and t are, to leading order, constant in the region just outside the hold region, we have from this differential equation that

$$f^4(z, \xi, t) = \left[b f_z^0(z, t)\right](\xi + \tilde{\gamma}) + f^4(z, -\tilde{\gamma}, t) \tag{127}$$

to leading order in the buy region near the hold region. From (52) and (54), we see that this linear function in ξ agrees at the buy boundary $\xi = -\tilde{\gamma}$ with the value of f^4, the derivative f_ξ^4, and the second derivative $f_{\xi\xi}^4$ of the solution inside the hold region. Similarly, we have also have agreement at the sell boundary, $\xi = \gamma$, with

$$f^4(z, \xi, t) = -\left[c f_z^0(z, t)\right](\xi - \tilde{\gamma}) + f^4(z, \tilde{\gamma}, t), \tag{128}$$

which is the leading order expression for f^4 in the sell region near the hold region. Therefore, the leading order of the solution is C^2 smooth. We note in the literature that agreement of the second derivative at the boundary is often called the "smooth pasting" condition (see, for example, [12], [13]).

With the single stock case in mind, we now return to the case of multiple uncorrelated stocks. Again we consider the area just outside the hold region. As in the single stock case, if we are in the trading region, we immediately trade so that the portfolio moves to the closest point on the boundary of the hold region. Specifically, if we are at ξ in the trading region, we move to the point ξ' on the boundary whose components are defined by

$$\xi'_i = \begin{cases} -\tilde{\gamma}_i & \text{if } \xi_i < -\tilde{\gamma}_i \\ \xi_i & \text{if } -\tilde{\gamma}_i \le \xi_i \le \tilde{\gamma}_i \\ \tilde{\gamma}_i & \text{if } \tilde{\gamma}_i < \xi_i. \end{cases}$$

(So $\xi_i^{'}$ is a function solely of ξ_i.) We define Δ_i, the leading order loss in expected utility from transacting stock i in this immediate trade to the boundary, by

$$\Delta_i = \begin{cases} b_i f_z^0(z,t)(-\xi_i - \tilde{\gamma}_i) & \text{if } \xi_i < -\tilde{\gamma}_i \\ 0 & \text{if } -\tilde{\gamma}_i \le \xi_i \le \tilde{\gamma}_i \\ c_i f_z^0(z,t)(\xi_i - \tilde{\gamma}_i) & \text{if } \tilde{\gamma}_i < \xi_i, \end{cases}$$

and so we have that, to leading order, in the trading region near the boundary of the hold region

$$f^4(z,\xi,t) = f^4(z,\xi',t) - \Delta_i. \tag{129}$$

Note that (129) reduces to (127) and (128) in the case of a single stock. From (129), we differentiate to determine the first derivatives in the trading region

$$f^4_{\xi_i}(z,\xi,t) = f^4_{\xi_i}(z,\xi',t)\mathbf{1}_{\xi_i \in [-\tilde{\gamma}_i, \tilde{\gamma}_i]} - \frac{d\Delta_i}{d\xi_i} = \begin{cases} b_i f_z^0(z,t) & \text{if } \xi_i < -\tilde{\gamma}_i \\ f^4_{\xi'_i}(z,\xi',t) & \text{if } -\tilde{\gamma}_i \le \xi_i \le \tilde{\gamma}_i \\ -c_i f_z^0(z,t) & \text{if } \tilde{\gamma}_i < \xi_i \end{cases} \tag{130}$$

where the indicator function $\mathbf{1}_\alpha$ is defined to equal 1 inside the set α and equal to 0 outside the set. From the first order boundary conditions, (110), we have that $f^4_{\xi'_i}(z,\xi',t) = b_i f_z^0(z,t)$ when $\xi_i = -\tilde{\gamma}_i$ and $f^4_{\xi'_i}(z,\xi',t) = -c_i f_z^0(z,t)$ when $\xi_i = \tilde{\gamma}_i$, so, since the solution is smooth within the hold region, we have that the first derivatives are, to leading order, continuous within a neighborhood of the hold region (that is, both inside and just outside the boundary of the hold region). Differentiating (130), we obtain the second derivatives

$$f^4_{\xi_i\xi_j}(z,\xi,t) = f^4_{\xi_i\xi_j}(z,\xi',t)\mathbf{1}_{\xi_i\in[-\tilde{\gamma}_i,\tilde{\gamma}_i]}\mathbf{1}_{\xi_j\in[-\tilde{\gamma}_j,\tilde{\gamma}_j]}. \tag{131}$$

From (125), the solution in the hold region and its boundary, we see that the mixed partials $f^4_{\xi_i\xi_j} = 0$ when $i \neq j$ and so, by (131), the mixed partials also equal zero in the trading region. The second order boundary condition (111) can be rewritten as $f^4_{\xi_i\xi_i}(z,\xi',t) = 0$ when $\xi'_i = \pm\tilde{\gamma}_i$ so from (131) and the smoothness of the solution in the hold region and its boundary, we have that $f^4_{\xi_i\xi_i}$ is continuous in the trading region near — and on — the boundary of the hold region. Therefore, all of the elements of the Hessian matrix are continuous in a neighborhood of the boundary of the hold region. Since, from (129), we have that the solution in the trading region is continuous near (and on) the boundary, we conclude that f^4 is C^2 smooth in ξ — that is, f^4 is a classical solution — within a neighborhood of the hold region.

3.6 Density Blow-up in the Uncorrelated Two Stock Case

In this section we show that for two uncorrelated stocks, the classical solution for f^4 from the previous section will generally correspond to a density function, u, in the rectangular hold region that becomes singular at two of the hold region's four corners. This, of course, is in marked contrast to the single stock case where u was shown to be uniform. Since we apply the equations of Section 3.2 in this section, we again revert to Section 3.2 and 3.4's scaling for ξ.

Our conclusions will apply the results of Trefethen and Williams (see section 3 of [11]) for probability functions governed by Laplace's equation within a polygonal domain subject to oblique boundary conditions. We begin by recalling our forward equation formulation (93), (100), and (101):

$$0 = A_{ij}u_{\xi_i\xi_j} \qquad \xi \in \mathcal{H},$$

$$\begin{aligned} 0 &= A_{ij}\left[u_{\xi_i}\left(\beta_k\right)_{\xi_j} + d_{\xi_i}\left(u\left(\beta_k\right)_{\xi_j}\right)\right] &\qquad \text{on } B_k \\ 0 &= -A_{ij}\left[u_{\xi_i}\left(\gamma_k\right)_{\xi_j} + d_{\xi_i}\left(u\left(\gamma_k\right)_{\xi_j}\right)\right] &\qquad \text{on } \Gamma_k, \end{aligned}$$

where we do not sum on k, and

$$\int_{\mathcal{H}} u(\xi)d\xi = 1.$$

Also, recall that A_{ij} was defined in (89) by

$$A_{ij} = \frac{1}{2} a_{ik} a_{jk}$$

and a_{ij} was defined in (86) by

$$a_{ij} = m_i \sigma_{ij} - (m_i)_z \, m_k \sigma_{kj}.$$

Since Trefethen and Williams' work applies to Laplace's equation, we will assume the case that the matrix $[a_{ij}]$ is invertible so we can define the variable transformation

$$\eta_i = a_{ij}^{-1} \xi_j \tag{132}$$

which transforms the above interior equation (93) from the forward equation formulation into Laplace's equation

$$0 = u_{\eta_i \eta_i}.$$

Note that for the uncorrelated case, a_{ij} reduces to

$$a_{ij} = m_i \delta_{ij} \sigma_j - (m_i)_z \, m_j \sigma_j.$$

While the first term in this expression is strictly diagonal, the second term, in general, is not. Therefore, for two stocks, the variable transformation (132) transforms the rectangular hold region, $\mathcal{H}$, in ξ-space into, in general, a *nonrectangular* parallelogram, $\mathcal{H}^\eta$, in η-space (see Figure 2).

The transformed normalization condition $\int_{\mathcal{H}^\eta} u(\eta) d\eta = 1/\det\left([a_{ij}]\right)$ is not as interesting as the transformed oblique boundary conditions. When transformed into η-space, the oblique conditions (100) take the form

$$\begin{aligned} 0 &= -a_{ki} \left[2u_{\eta_i} - a_{jk}^{-1} a_{ki} u_{\eta_j} \right] \qquad \text{on } B_k^\eta \\ 0 &= a_{ki} \left[2u_{\eta_i} - a_{jk}^{-1} a_{ki} u_{\eta_j} \right] \qquad \text{on } \Gamma_k^\eta, \end{aligned}$$

where we do not sum over k. A long, but straightforward, calculation reconfirms a statement in Williams and Trefethen, that for two stocks, this new oblique condition has a remarkably simple geometric interpretation in η-space: Namely, on the boundary, the derivative of u is zero in a direction that forms an angle, θ, to the boundary, where θ is the acute angle formed by the parallelogram's sides. We also note that if this direction where the

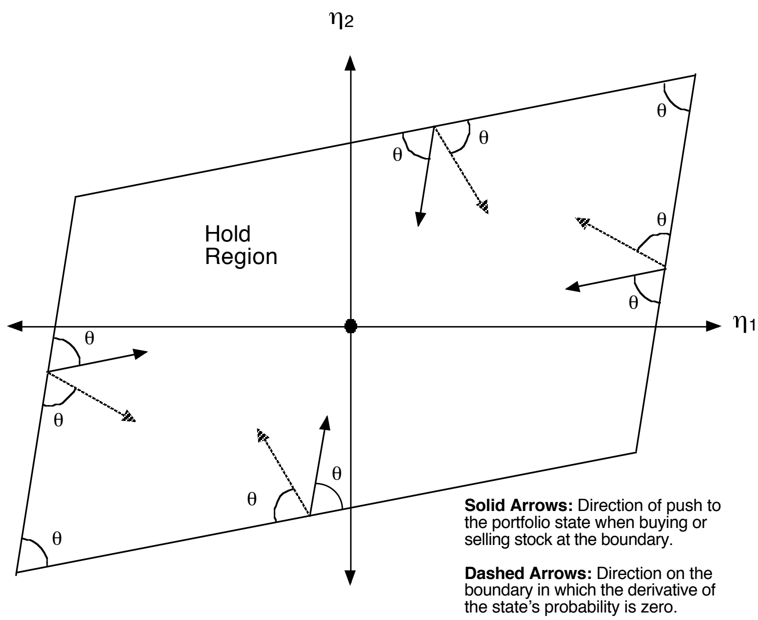

Figure 2: *The Hold Region for two uncorrelated stocks in the η coordinate system.*

derivative of u equals zero is flipped about the normal to the boundary, it corresponds to the direction of push due to stock sales or purchases at the boundary. Again, we refer the reader to Figure 2.

Restricting the results from Trefethen and Williams to the case of a parallelogram, we have that u is constant only when the parallelogram in η-space is a rectangle; that is, only when $\theta = \frac{\pi}{2}$. For any other angle, $\theta \neq \frac{\pi}{2}$, the density becomes infinite (but remains integrable, of course) as the two obtuse vertices of the parallelogram are approached, while the density approaches zero as the two acute vertices are approached. At a simple level, this can be understood as the vectors for purchasing and selling stock at the boundary push the state of the portfolio into the obtuse corners and away from the acute corners. For a better understanding, along with determining the asymptotic rate at which the density becomes infinite or goes to zero, we magnify our view of the vertex so that near the vertex, we can approximate the problem as acting on an infinite wedge whose corner is the vertex. A solution in polar coordinates to Laplace's equation on the wedge is

$$u(r, \phi) = Cr^m \cos\left(m(\phi + \phi_1)\right)$$

for constants C, m, and ϕ_1. The oblique boundary conditions yield, for some constant $\phi_{2,}$ that $\cos\left(\theta - m\phi_2\right) = \cos\left(\theta + m(\theta + \phi_2)\right) = 0$, which can be combined to obtain $2\theta + m\theta = n\pi$ for some integer n. When $\theta = \frac{\pi}{2}$, there is no push into the vertices from buying or selling, so it follow that u is constant and therefore, $m = 0$. Inserting this fact into $2\theta + m\theta = n\pi$, yields that $n = 1$ and therefore, we have that m, the exponent for the rate of blow-up or disappearance of u at the vertex is

$$m = \frac{\pi}{\theta} - 2.$$

For $\theta < \frac{\pi}{2}$, we have that $m > 0$ so this gives the rate at which u decays to zero we approach the vertex. This formula also holds for the obtuse angles of the parallelogram so, for $\frac{\pi}{2} < \theta < \pi$, we have that $-1 < m < 0$, which gives the rate of blow-up. Note also that since $-1 < m$, we have that the density remains integrable near the vertex.

4 Appendices

4.1 Appendix 1: Second derivative boundary condition

In this appendix we use the Calculus of Variations to show how the first variation of the boundary leads to a second derivative boundary condition.

Let $O[\cdot]$ be a given differential operator so that $f_t + O[f] = 0$ is a partial diffential equation for $f(x,t)$ subject to the maximum principle, as is the case for the parabolic equations encountered in this paper. Let $x \in \Omega(t) \subset \mathbf{R}^n$ where $\partial\Omega(t)$ is piecewise smooth, $\Omega(t)$ is continuous in $t \in [0,T]$, and, most importantly, $\Omega(t)$ is defined so that $\Omega^T = \bigcup_{t\in[0,T]} (\Omega(t), t)$ is the region in $\mathbf{R}^n \times [0,T]$ that optimizes the solution, $f(x,t)$, to our partial differential equation

$$f_t + O[f] = 0 \qquad \text{for } (x,t) \in \Omega^T \tag{133}$$

subject to a given initial condition and the boundary condition

$$v(x,t) \cdot \nabla_x f(x,t) = 0 \qquad \text{for } x \in \partial\Omega(t). \tag{134}$$

Now consider a parameterized mapping $M_\varepsilon : \Omega(t) \to \Omega_\varepsilon(t)$ from the optimal region to suboptimal regions when $\varepsilon \neq 0$ defined by

$$x \mapsto x + \varepsilon w(x,t) \tag{135}$$

where $w(x,t) \in \mathbf{R}^n$ is a fixed, but arbitrary, vector field, subject to $w(x,0) = 0$ so that the initial condition is unchanged by ε. On $\Omega_\varepsilon^T = \bigcup_{t\in[0,T]} (\Omega_\varepsilon(t), t)$ we have that the suboptimal solutions, $f(x,t,\varepsilon)$, still satisfy the partial differential equation (133), the initial condition, and the boundary condition

$$v(x,t) \cdot \nabla_x f(x,t,\varepsilon) = 0 \qquad \text{for } x \in \partial\Omega_\varepsilon(t).$$

Taking the derivative of the boundary condition with respect to ε at locations where $\partial\Omega_\varepsilon(t)$ is smooth, we obtain

$$\nabla_x \left(v(x,t) \cdot \nabla_x f(x,t,\varepsilon) \right) \cdot \frac{\partial x}{\partial \varepsilon} + v(x,t) \cdot \nabla_x \frac{\partial f}{\partial \varepsilon} = 0 \qquad \text{for } x \in \partial\Omega_\varepsilon. \tag{136}$$

Noting from (135) that ε affects x on the boundary through the relation $x(\varepsilon) = x + \varepsilon w(x,t)$, we have that $\frac{\partial x}{\partial \varepsilon} = w(x,t)$, and since f is optimized at

$\varepsilon = 0$, from the maximum principle, we have that $\left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$; therefore, at $\varepsilon = 0$, (136) reduces to

$$\nabla_x \left(v(x,t) \cdot \nabla_x f(x,t,0) \right) \cdot w(x,t) = 0 \qquad \text{for } x \in \partial\Omega_0. \tag{137}$$

Since w is arbitrary and the optimal region $\Omega = \Omega_0$, we see from (137), that the optimality of Ω has generated the second derivative boundary condition

$$\nabla_x \left(v(x,t) \cdot \nabla_x f(x,t) \right) = 0 \qquad \text{for } x \in \partial\Omega. \tag{138}$$

Note, however, that the first derivative condition (134) already implies that the directional derivative of $v(x,t) \cdot \nabla_x f(x,t)$ is zero in any direction tangent to $\partial\Omega$, so the only new implication of (138) is that the directional derivative of $v(x,t) \cdot \nabla_x f(x,t)$ is zero in any direction that is *transverse* to the boundary. This means that while (138) contains n conditions, only one condition is not immediately implied by the first derivative condition (134).

4.2 Appendix 2: abstract control duality

For completeness we discuss the abstract adjoint formulation which leads to the adjoint equation in the body of the paper for the optimal shape of the holding region, $\mathcal{H}$. The ideas presented here are not new but our treatment of boundary conditions and our derivation of boundary conditions for the adjoint problem are not completely standard. We will develop this appendix with a view towards deriving adjoint PDEs by treating the original PDE as a collection of constraints, one applied at each point in the region Ω where the PDE should be satisfied (so Ω is $\mathcal{H}$ in the body of this paper). The adjoint field, which satisfies the adjoint PDE, will be thought of as a collection of Lagrange multipliers, one for each $x \in \Omega$. We also use this point of view to introduce an additional Lagrange multiplier field defined for each $x \in \partial\Omega$, which leads to an additional boundary condition for the adjoint PDE when conditions on $\partial\Omega$ are optimized. To motivate these PDE constructions, we begin with a simple discussion of the situation in which the infinite dimensional constraints of the PDE and its boundary conditions are replaced by a finite set of m constraints and in which the optimal shape Ω for our expected utility is replaced by an optimal finite dimensional vector $v \in \mathbf{R}^n$.

In elementary Lagrange multiplier problems, we look to optimize a scalar function $f(v)$ subject to m constraints, $g(v) = 0$, (so $g \in \mathbf{R}^m$) by finding a row vector of Lagrange multipliers, $\lambda \in \mathbf{R}^n$, and values of v where

$$0 = f_v - \lambda g_v. \tag{139}$$

Here g_v represents the matrix of mixed partials, $\frac{\partial g_i}{\partial v_j}$, and f_v represents the row vector, $\frac{\partial f}{\partial v_j}$.

We now change our problem to optimize the scalar function $f(v, u)$ subject to m constraints, $g(v, u) = 0$, where $u \in \mathbf{R}^m$. Since both g and u are in $\mathbf{R}^m$, if we assume — as we now do — the $m \times m$ matrix, g_u, is invertible, then the constraints, along the implicit function theorem, imply that u, locally, is a function of our optimal control variables, v. Therefore, we can calculate $d_v g$, the $(m \times n)$ total derivative of $g(v, u(v))$ with respect to v,

$$0 = d_v g = g_v + g_u u_v,$$

to obtain the Jacobian matrix, u_v :

$$u_v = -(g_u)^{-1} g_v. \tag{140}$$

Next we look for v that optimize f by finding where the total derivative, $d_v f$, is zero, and, after applying our Jacobian matrix formula (140), we have

$$\begin{aligned} 0 &= d_v f = f_v + f_u u_v \\ &= f_v - f_u (g_u)^{-1} g_v. \end{aligned} \tag{141}$$

Matching (141) with (139), we see that the Lagrange multipliers in our current context are $\lambda = f_u (g_u)^{-1}$, or

$$\lambda g_u = f_u. \tag{142}$$

Note that if we take the transpose (that is, the adjoint) of (142), we get an equation for the column vector λ^T. This is why (142) is called the *adjoint* equation.

In Appendix 1, we considered one parameter alterations of the optimal boundary points, x; that is $x(\varepsilon) = x + \varepsilon w$ (see equation (135)). For this appendix, v, as opposed to x, corresponds to conditions on the boundary, so we now consider an optimization problem with control variables of the form

$$v(\varepsilon) = v_0 + \varepsilon \dot{v}$$

where v_0 are specific values that optimize f. Notice that w, the arbitrary vector of perturbation in (135), is now denoted by $\dot{v}$. This is consistent with our now defining the dot to represent the derivative of a quantity with respect to ε evaluated at $\varepsilon = 0$. For example, in the case at hand for v, we have $\dot{v} = \frac{dv}{d\varepsilon}\Big|_{\varepsilon=0}$.

Although u is still a function of v, which is now a function of ε, we will suppress this dependence and just write $u(\varepsilon)$, so our new problem is to optimize the scalar function $f(v(\varepsilon), u(\varepsilon))$ subject to the m constraints, $g(v(\varepsilon), u(\varepsilon)) = 0$. As before, we compute the total derivative of the constraints,

$$0 = \dot{g} = g_v\dot{v} + g_u\dot{u},$$

to obtain the column vector, $\dot{u}$:

$$\dot{u} = -(g_u)^{-1}g_v\dot{v}. \tag{143}$$

Since $v(0) = v_0$ is optimal for f, we have, as before,

$$\begin{aligned} 0 &= \dot{f} = f_v\dot{v} + f_u\dot{u} \\ &= f_v\dot{v} - f_u(g_u)^{-1}g_v\dot{v} \\ &= (f_v - \lambda g_v)\,\dot{v} \end{aligned} \tag{144}$$

where λ is defined, as before, by (142).

In preparation for the PDE example to follow, we need to multiply the Lagrange multiplier equation, (142), by $\dot{u}$,

$$\lambda g_u\dot{u} = f_u\dot{u} \tag{145}$$

where $\dot{u}$ now represents any arbitrary perturbation at $\varepsilon = 0$, not just the $\dot{u}$ perturbation defined in (143). Notice that $g_u\dot{u}$ is $\frac{d}{d\varepsilon}g(v, u(\varepsilon))\Big|_{\varepsilon=0}$, the derivative *if we freeze* v evaluated at $\varepsilon = 0$. An analogous statement holds for $f_u\dot{u}$.

So, to summarize, we begin with the expression

$$f - \lambda g. \tag{146}$$

We set the perturbation of (146) with respect to u while holding v constant equal to zero, which yields (145), the Lagrange multiplier equation

$$f_u\dot{u} = \lambda g_u\dot{u}, \tag{147}$$

and then, by optimality, we also set the perturbation of (146) with respect to v while holding u constant equal to zero, which yields (144), the optimality equation

$$f_v \dot{v} = \lambda g_v \dot{v}. \tag{148}$$

Now we consider a simple model problem to illustrate how the above ideas generalize to a PDE optimization problem. Let Ω be a bounded region with a smooth boundary $\partial\Omega$. For a given function, $\phi(x)$, we seek to minimize

$$\frac{1}{2}\int_{\Omega} (u(x) - \phi(x))^2 \, dx \tag{149}$$

subject to the constraints that 1) the partial differential equation

$$\triangle u + \frac{1}{2}u^2 = 0 \qquad \text{for } x \in \Omega \tag{150}$$

and 2) the boundary condition

$$u(x) = v(x) \qquad \text{for } x \in \partial\Omega \tag{151}$$

are satisfied. Note, as before, that once we choose $v(x)$ on the boundary, the constraints, g, given in (150) and (151), define what $u(x)$ must be throughout Ω, so our goal is to find optimal boundary values $v(x)$ that lead to the optimal (in this case minimal) $f(u)$ given in (149).

Since one of our constraints (150) is defined for all $x \in \Omega$ and the other (151) is defined for $x \in \partial\Omega$, we must consider separate Lagrange multiplier fields for each. Specifically, λ in our previous context is now replaced by $\lambda(x)$ for the $x \in \Omega$ corresponding to the PDE constraint (150) and by $\mu(x)$ for the $x \in \partial\Omega$ corresponding to the boundary constraint (151). Also, the inner products in the Lagrange multiplier equation (147) were defined in our previous finite context by matrix multiplication. These inner products now become integration over Ω or $\partial\Omega$. Recalling that $g_u \dot{u}$ and $f_u \dot{u}$ are derivatives with respect to ε while freezing v, (147) now takes the form

$$\begin{aligned} & \int_{\Omega} (u(x) - \phi(x))\, \dot{u}(x) dx \\ = \; & \int_{\Omega} \lambda(x) \left(\triangle \dot{u}(x) + u(x)\dot{u}(x)\right) dx + \int_{\partial\Omega} \mu(x)\dot{u}(x) dx. \end{aligned}$$

We must choose $\lambda(x)$ and $\mu(x)$ so that this holds for any perturbation $\dot{u}$. The standard calculus of variations way to insure this is to integrate by parts over Ω using Green's formula to get (with ∂_n representing differentiation in the outward normal direction)

$$\int_\Omega (u(x) - \phi(x))\dot{u}(x)dx$$
$$= \int_\Omega (\triangle\lambda(x) + \lambda(x)u(x))\dot{u}(x)dx + \int_{\partial\Omega} \lambda(x)\partial_n\dot{u}(x)dA(x)$$
$$- \int_{\partial\Omega} \partial_n\lambda(x)\dot{u}(x)dA(x) + \int_{\partial\Omega} \mu(x)\dot{u}(x)dA(x) \; .$$

The calculus of variations argument shows that the coefficient of $\dot{u}$ must vanish identically. In the interior of Ω, this implies the following PDE for λ

$$u - \phi = \triangle\lambda + \lambda u \; . \tag{152}$$

Similarly, if we think of $\dot{u}$ and $\partial_n\dot{u}$ as different variables on the boundary and set their coefficients equal to zero separately, we get

$$\partial_n\lambda(x) = \mu(x) \quad \text{for all } x \in \partial\Omega \; , \tag{153}$$

and

$$\lambda(x) = 0 \quad \text{for all } x \in \partial\Omega \; . \tag{154}$$

We now have our adjoint PDE (152) and adjoint boundary conditions (154). To understand the role of condition (153), we consider the optimization formula (148). Since the control variables, v, appear in the boundary constraint (151), but not in the PDE constraint (150) nor in the function (149) to be minimized, the only term in (144) that remains is the term corresponding to the inner product of μ with $g_v\dot{v}$, so (148) becomes

$$0 = \int_{\partial\Omega} \mu(x)\dot{v}(x)dx.$$

Since $\dot{v}(x)$ is arbitrary, we have that $\mu(x) = 0$, and, so by (153), we have that

$$\partial_n\lambda(x) = 0. \tag{155}$$

In other words, when v is optimal, it corresponds to a function u where the solution to the adjoint PDE system (152) and (154) also satisfies the boundary condition (155).

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