Lecture 6

Now as \( \frac{d}{dt} \alpha_i(t) = 2b_i(t) + \gamma t \), we

see that \( \alpha_i(t) \) is monotone decreasing as \( t \to -\infty \).

That is, as above,

\[
\alpha_i(-\infty) = \lim_{t \to -\infty} \alpha_i(t)
\]

exist and

\[
b_i(t) \to 0 \quad \text{as} \quad t \to -\infty.
\]

Continuing we find that

\[
a_k(-\infty) = \lim_{t \to -\infty} a_k(t), \quad k = 1, \ldots, N
\]

exist and

\[
b_k(t) \to 0 \quad \text{as} \quad t \to -\infty, \quad k = 1, \ldots, N-1.
\]

Necessary as \( a_k(-\infty) \)'s are the eigenvalues of \( X(0) \).

Arguing as above, we have for \( k = 1, \ldots, N \)

\[
x_k(t) = -2a_k(-\infty)t + o(t) \quad \text{as} \quad t \to -\infty
\]

and so for \( k = 1, \ldots, N-1 \)

\[
x_k(t) - x_{k+1}(t) = -2(a_k(-\infty) - a_{k+1}(-\infty))t + o(t)
\]

but again as \( b_k = \frac{1}{2} e^{\frac{1}{2}(x_k(t) - x_{k+1}(t))} \to 0 \) as \( t \to -\infty \),

we must have
\[ a_k(-\infty) - a_{k+N}(-\infty) < 0 \]

It follows that we must have

\[ a_k(-\infty) \to \lambda_{N-k+1}, \quad 1 \leq k \leq N \]

Thus we have a billiard ball type interaction as

\[ t \to -\infty, \]

\[ \begin{array}{cccc}
-2\lambda_1 & -2\lambda_2 & \cdots & -2\lambda_N \\
-x_1 & -x_2 & \cdots & -x_N \\
x_1 & x_2 & \cdots & x_N \\
\end{array} \]

and as \( t \to \infty \)

\[ \begin{array}{cccc}
-2\lambda_1 & -2\lambda_2 & \cdots & -2\lambda_N \\
-x_1 & -x_2 & \cdots & -x_N \\
x_1 & x_2 & \cdots & x_N \\
\end{array} \]

So the particle \( x_N \), which at velocity \( -2\lambda_1 \) as \( t \to -\infty \), transfers its velocity to \( x_1 \) as \( t \to +\infty \), etc. This is reminiscent of a ball impinging on a row of balls with velocity \( v \), and transferring its velocity to the end ball after collision.

Finally observe that from (93.11) and (95.1), \( b_k(t) \to \) exponential.
We now compute the error term \( \alpha(t) \) in (9.1.1) more precisely. This requires a more detailed approach.

From the theory of tensors we have for vectors \( u_0, u_1, \ldots, u_k \) in \( \mathbb{R}^n \):

\[
(u_0 \wedge u_1 \wedge \ldots \wedge u_k, u_0 \wedge u_1 \wedge \ldots \wedge u_k)
\]

\[
= \det \begin{pmatrix} (u_i, u_j) \end{pmatrix}_{0 \leq i, j \leq k}.
\]

In particular for

\[
u_i = x_0^k e_i, \quad 0 \leq i \leq n
\]

where \( x_0 = x(0) \) and \( e_i = (1, 0, \ldots, 0)^T \)

\[
(e_1 \wedge x_0 e_1 \wedge \ldots \wedge x_0^k e_1, e_1 x_0 e_1 \wedge \ldots \wedge x_0^k e_1)
\]

\[
= \det \begin{pmatrix} (e_1, x_0 e_1) & (e_1, x_0^2 e_1) & \cdots & (e_1, x_0^k e_1) \\
(x_0 e_1, x_0 e_1) & (x_0^2 e_1, x_0 e_1) & \cdots & (x_0^k e_1, x_0 e_1) \\
(x_0^k e_1, e_1) & (x_0^{k+1} e_1, e_1) \\
(x_0^{k+1} e_1, x_0^k e_1) & (x_0^{k+1} e_1, x_0^{k+1} e_1) 
\end{pmatrix}
\]
\[ \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & & \\ \vdots & & & \vdots \\ c_n & & & c_0 \end{pmatrix} = 0_h, \]

where

\[(98.1) \quad c_j = (x_0^j e_i, e_i) = \sum_{i=1}^{N} \lambda_i^j \psi_{i}^{(1)}, \]

where \(\lambda_i\)'s are the eigenvalues of \(X_0\) and \(u_i^{(1)} > 0\) are the first components of \(X_0\) corresponding eigen vectors.

Recollecting, we have

\[(98.2) \quad c_i = \int \lambda^i \psi_{\lambda}(\lambda) \]

where

\[(98.3) \quad \psi_{\lambda}(\lambda) = \sum_{i=1}^{N} u_i^{(1)} \delta_{\lambda i}(\lambda) \]

On the other hand as \(X_0 = \begin{pmatrix} a_1 & b_1 & \cdots & 0 \\ b_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \) is a Jacobian matrix,

\[X_0 e_i = b_i e_i + a_i e_i = b_i e_i + \gamma_i, \quad \gamma_i \in \mathbb{R} \]

\[X_0^2 e_i = b_i X_0 e_i + a_i X_0 e_i = b_i b_i e_i + \gamma_i, \quad \gamma_i \in \mathbb{R} \]

and by induction

\[(98.4) \quad X_0^k = b_1 b_2 \cdots b_k e_i, + \gamma_k, \quad \text{where} \quad \gamma_k \in \mathbb{R} \]
Thus
\[ c_1 \wedge x_0 e_1 \wedge \ldots \wedge x_0^k c_1. \]

\[ = c_1 \wedge (b_1 e_1 + r_1) \wedge (b_1 b_2 e_2 + r_2) \wedge \ldots \wedge (b_1 \ldots b_k e_{k+1} + r_k) \]

\[ = b_1 (c_1 \wedge e_1 \wedge (b_1 b_2 e_2 + r_2) \wedge \ldots \wedge (b_1 \ldots b_k e_{k+1} + r_k) \]

\[ = b_1 (b_1 b_2) (b_1 b_2 b_3) \ldots (b_1 \ldots b_k) (c_1 \wedge e_1 \wedge \ldots \wedge e_k) \]

Thus

\[ D_k = b_1^k (b_1 b_2)^{k-1} \ldots (b_1 \ldots b_k)^{1} \]

\[ \text{for } 1 \leq k \leq N-1, \]

and

\[ D_0 = 1 \]

Hence for \( 1 \leq k \leq N-1 \)

\[ \frac{D_k}{D_{k-1}} = \frac{b_1^k (b_1 b_2)^{k-1} \ldots (b_1 \ldots b_k)^{1}}{b_1^{k-1} (b_1 b_2)^{k-2} \ldots (b_1 \ldots b_{k-1})^{1} \ldots (b_1 \ldots b_k)^{1}} \]

\[ = \frac{(b_1 \ldots b_{k-1})^1 (b_1 \ldots b_k)^1}{(b_1 \ldots b_{k-1})^{k-2}} \]

\[ = b_k \]

Also for \( k = 2 \)

\[ \frac{D_2}{D_1} = \frac{b_1^2 (b_1 b_2)^{1}}{b_1^{1}} = b_2 \]

\[ \text{and for } k = 1 \]

\[ \frac{D_1}{D_0} = b_1^1 \]
\[ \Delta_i = 1 \]

Thus

\[ \delta_k^2 = \frac{\Delta_{k-1}}{\Delta_k}, \quad 1 \leq k \leq N-1, \]

provided \( \Delta_{-1} = 1 \).

Note from (98.1) that (100.1) expresses \( b_k \), and hence \( x_k - x_{k-1} \), directly in terms of the data \((x_1, \ldots, x_N)\) and \((u_1(1), \ldots, u_k(1))\), whose evolution under the isospectrality and is known explicitly from (86.5).

Now we expand \( \Delta_k \) into a more useful formula as follows: By (98.3) and the formula for a Vandermonde determinant,

\[
\Delta_k = \begin{vmatrix}
\int x_k^0 q_1(x_0) & \int x_0 x_k q_1(x) & \cdots & \int x_k^0 x_k q_k(x_k) \\
\int x_0^0 q_1(x_0) & \int x_0 x_1 q_1(x) & \cdots & \int x_1^0 x_k q_k(x_k) \\
\vdots & \vdots & \ddots & \vdots \\
\int x_k^0 q_1(x_0) & \int x_0 x_k q_1(x) & \cdots & \int x_k^0 x_k q_k(x_k)
\end{vmatrix}
\]

\[
= \int - \int q_1(x_0) \cdots q_k(x_k) \text{ det} \begin{pmatrix}
1 & x_1^k & \cdots & x_k^k \\
x_0 & x_1 & \cdots & x_k \\
x_0^k & x_1^k & \cdots & x_k^k \\
x_0 & x_1 & \cdots & x_k
\end{pmatrix}
\]
\[= \sum_{k=0}^{\infty} \frac{\phi(u(x_0)) \phi(u(x_k)) x_0^k}{k!} \int_0^1 \chi_0^{x_0} \chi_1^{x_1} \cdots \chi_k^{x_k} \mu(x_0) \mu(x_1) \cdots \mu(x_k) dx_0 dx_1 \cdots dx_k\]

Let \( \Pi \) be the permutation group acting on \( 0, 1, \ldots, k \), we find

\[D_k = \sum_{\Pi} \sum_{(k+1)!} \phi(u(x_{\Pi(0)}) \cdots \phi(u(x_{\Pi(k)})) x_0^{\Pi(0)} \cdots x_k^{\Pi(k)} \cdot (\text{action of } \Pi \text{ on } x_{\Pi(k)})\]

But the action of \( \Pi \) on the Vandermonde determinant is given by

\[\det \left( \begin{array}{cccc}
\chi_{\Pi(0)} & \chi_{\Pi(1)} & \cdots & \chi_{\Pi(k)} \\
\chi_{\Pi(j)} & \chi_{\Pi(j+1)} & \cdots & \chi_{\Pi(j+k)} \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{\Pi(k(j-1))} & \chi_{\Pi(k(j-1)+1)} & \cdots & \chi_{\Pi(k(j-1)+k)}
\end{array} \right) = \text{sgn } \Pi \cdot \det \left( \begin{array}{cccc}
\chi_{0} & \chi_{1} & \cdots & \chi_{k} \\
\chi_{0} & \chi_{1} & \cdots & \chi_{k} \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{0} & \chi_{1} & \cdots & \chi_{k}
\end{array} \right)\]

Thus

\[D_k = \frac{1}{(k+1)!} \sum_{\Pi} \phi(u(x_{\Pi(0)}) \cdots \phi(u(x_{\Pi(k)})) x_0^{\Pi(0)} x_1^{\Pi(1)} \cdots x_k^{\Pi(k)} \cdot \text{sgn } \Pi \cdot \prod_{i<j}^{k} (x_i - x_j)
\]
\[ = \frac{1}{(k+1)!} \int \int \int \cdots \int \phi(x_0) \cdots \phi(x_k) \prod_{0 \leq i < j \leq k} (x_j - x_i)^2 \] 

for \( k \geq 1 \)

\[(10.2.1) \quad \Theta_k = \frac{1}{(k+1)!} \int \int \int \cdots \int \phi(x_0)x_0 \cdots \phi(x_k)x_k \prod_{0 \leq i < j \leq k} (x_j - x_i)^2 \]

as \( \prod (x_j - x_i)^2 \) is invariant under permutations and terms with \( x_i = x_j \) for some \( i \) just drop out. As before \( D_k \) is.

Now under the Toda flow

\[(10.2.3) \quad \phi_t(x,t) = \sum_{c} u_c^2(1,\xi) \delta_{x,c}(\xi) \]

\[= \sum_{c} u_c^2(1,\xi) e^{2 \lambda_i t} \delta_{x,c}(\xi) \]

\[= \sum_{c} u_c^2(1,\xi) e^{2 \lambda_i t} \frac{e^{-x^2} \phi(x)}{\int e^{-x^2} \phi(x)} \]

where \( \phi(x) \) is given by \( (9.8.3) \)

Thus under the Toda flow, for \( k \geq 1 \)
\[(03.1)\quad \Theta_h \left( t \right) = \frac{1}{\left( \int e^{2x^T \phi(x)} \right)^{k+1}} \int \int \ldots \int \sum_{0 \leq c1 \leq h} e^{2(\sum_{j=c1}^{c_{k-1}} x_j + c_{k-1} x_k) \phi(x)} \prod_{i=1}^{k} \left( \prod_{j=c_{i-1}+1}^{c_i} \right)\]

\[\text{and } \mu \text{ from (100.1), for } h \geq 2,\]

\[(03.2)\quad b_h = \left( \int e^{2x^T \phi(x)} \right)^{-(k+1) - (k-1) + 2h} \times \left( \int \ldots \int \sum_{0 \leq c1 \leq h} e^{2(\sum_{j=c1}^{c_{k-1}} x_j + c_{k-1} x_k) \phi(x)} \prod_{i=1}^{k} \left( \prod_{j=c_{i-1}+1}^{c_i} \right) \right)\]

\[\times \int \ldots \int \sum_{0 \leq c1 \leq h-2} e^{2(\sum_{j=c1}^{c_{k-1}} x_j + c_{k-1} x_k) \phi(x)} \prod_{i=1}^{k-1} \left( \prod_{j=c_{i-1}+1}^{c_i} \right)\]

\[\times \left( \int \ldots \int \sum_{0 \leq c1 \leq h-1} e^{2(\sum_{j=c1}^{c_{k-2}} x_j + c_{k-2} x_k) \phi(x)} \prod_{i=1}^{k-2} \left( \prod_{j=c_{i-1}+1}^{c_i} \right) \right)^{-1}\]

Thus for \(h \geq 2\)

\[(03.3)\quad b_h^2 = \left( \int \ldots \int \sum_{x_0 \ldots x_k} e^{2(\sum_{j=0}^{k} x_j) \phi(x)} \prod_{i=1}^{k} \phi(x_i) \right) \left( \int \ldots \int \sum_{x_0 \ldots x_{k-2}} e^{2(\sum_{j=0}^{k-2} x_j + x_{k-2}) \phi(x)} \prod_{i=1}^{k-2} \phi(x_i) \right) \left( \int \ldots \int \sum_{x_0 \ldots x_{k-1}} e^{2(\sum_{j=0}^{k-1} x_j) \phi(x)} \prod_{i=1}^{k-1} \phi(x_i) \right)\]

\[\text{and } \mu \text{ from (100.1), for } h \geq 2,\]

\[(03.4)\quad b_h = \left( \int \ldots \int \sum_{x_0 \ldots x_k} e^{2(\sum_{j=0}^{k} x_j) \phi(x)} \prod_{i=1}^{k} \phi(x_i) \right) \left( \int \ldots \int \sum_{x_0 \ldots x_{k-2}} e^{2(\sum_{j=0}^{k-2} x_j + x_{k-2}) \phi(x)} \prod_{i=1}^{k-2} \phi(x_i) \right) \left( \int \ldots \int \sum_{x_0 \ldots x_{k-1}} e^{2(\sum_{j=0}^{k-1} x_j) \phi(x)} \prod_{i=1}^{k-1} \phi(x_i) \right)\]
Where

\[(104.1) \quad dm^h(x) = dm(x_1) \cdots dm(x_k) \quad \text{and} \quad V_k(x) = \text{van der Monde} = \prod_{i=1}^{k}(x_i - x_j), \quad \text{for} \quad 0 \leq k \leq n.
\]

For \( k = 2 \), from (100.1)

\[(104.2) \quad b^2 = \frac{\partial^2 \theta_2}{\partial \alpha^2} \theta_0 = \frac{\partial^2 \theta_2}{\partial \beta^2} = \left( e^{x_1} q(x_1) \right)^{-3} \left( e^{x_1} q(x_1) \right)^4
\]

\[
\int \frac{q(x_0)\theta_0(x_1) \theta_0(x_2) e^{2(x_0+x_1+x_2)} + 2 \theta_2(x_2)}{e^{x_0+x_2}} dx_1 dx_2
\]

\[
\left( \int e^{x_0+x_2} \frac{q(x_0)\theta_0(x_2) e^{2(x_0+x_1)} + 2 \theta_2(x_1)}{e^{x_0+x_1}} \right)^2
\]

and for \( k = 1 \)

\[(104.3) \quad b^1 = \frac{\partial \theta_1}{\partial \alpha} = \frac{1}{e^{x_1} q(x_1)} \int_{x_0}^{x_1} q(x_0)\theta_1(x_1) e^{2(x_0+x_1)} + 2 \theta_2(x_1) dx_1
\]

Expanding we obtain:

\[
b_1 = \frac{1}{2} \int q(x_0, x_1) q(x_1, x_2) (x_0 - x_1) dx_1
\]

\[
= \frac{1}{2} \left[ \int q(x_0, x_1) x_0^5 - 2 \left( \int q(x_1) \right) x_1^4 \right]
\]

\[
= \int x_1^2 q(x_1) + \left( \int q(x_1) \right)^2
\]

Recall

\[
b^2 = \sum_{i=1}^{N} a_i q_i(x_{1+}) + \sum_{i=1}^{M} \frac{u_i(x_{1+})}{x_{1+}} - \left( \sum_{i=1}^{N} u_i(x_{1+}) \right)^2
\]

which matches (14), but (104.1) yields the leading behavior
$b_i(t)$ directly. Indeed as $t \to \infty$, noting that $V_k(u) = 0$

(10.5.1) $b_i^k(t) = \left[ \frac{1}{(c^{2} \lambda^{2} + c^{2})^t} \right]^{1-e^{-2(c^{2}/\lambda_s)}}

\times \left( u_i^1(1) \cdot u_i^2(1) \cdot e^{2(c^{2}/\lambda_s - \lambda_s)} + \frac{c^{2}(\lambda_s - \lambda_s)}{(\lambda_s - \lambda_s)} \right) + O(e^{-c^{2}/\lambda_s})$

which agrees with (Eq. 3).}

Now for $k \geq 1$, we have from (10.5.1), as $t \to \infty$

$$b_k^2(t) = \left( u_i^1(1) \cdot u_k^1(1) \cdot e^{2(c^{2}/\lambda_s + \lambda_s) + t} \cdot \prod_{j \neq k} \left( \lambda_j = \lambda_s + \lambda_s \right) \right)$$

\[ \times \left( u_k^1(1) \cdot u_k^2(1) \cdot e^{2(c^{2}/\lambda_s + \lambda_s)} + \frac{c^{2}(\lambda_s + \lambda_s)}{(\lambda_s + \lambda_s)} \right) + O(e^{-c^{2}/\lambda_s}) \]

(10.5.1) $b_k^2(t) = \left( u_k^1(1) \cdot u_k^2(1) \cdot e^{2(c^{2}/\lambda_s + \lambda_s)} + \frac{c^{2}(\lambda_s + \lambda_s)}{(\lambda_s + \lambda_s)} \right) + O(e^{-c^{2}/\lambda_s})$

\[ \times \left( 1 + \exp \text{ small} \right) \]

\[ \left( 1 + \exp \text{ small} \right) \]
For $k = 2$, from (10.5.1) we have as $t \to \infty$,

\[ b_k(t) = \left( C^2 k^k u_1(t) + \text{exp. small} \right) \times \frac{u_1(t) u_2(t) u_2(t) e^{2(\lambda_1 + \lambda_2 + \lambda_3) \lambda} (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3) (\lambda_2 - \lambda_3)}{(u_1(t) u_2(t))^2 e^{2(\lambda_1 + \lambda_2) \lambda} (\lambda_1 - \lambda_2)^2} \times (1 + \text{exp. small}) \]

Thus we see that (10.5.1) also holds for $k = 2$, and

by (10.5.1) also holds for $k = 1$, provided we interpret

\[ (10.6.1) \quad \prod_{\lambda \neq 0} (\lambda - \lambda_{n+1})^2 = 1. \]

Exercise: Compute the analogous formulae for $a_k(t)$, $t \to \infty$.

A similar calculation shows that as $t \to -\infty$,

\[ b_k(t) = \frac{u_{N-k}(t)}{u_{N-k+1}(t)} \prod_{j=0}^{N-k} (\lambda_{N-k} - \lambda_j)^2 e^{2t (\lambda_{N-k} - \lambda_{N-k+1})} (1 + \text{exp. small}) \]
Again, this formula holds for $k = 1, \ldots, n-1$, provided we interpret the denominator in (106.1) as 1, i.e.

$$b_k^2(t) = \frac{u_{n-1}^2(t)}{u_n^2(t)} \left( \lambda_{n-1} - \lambda_n \right)^2 \left( 2^t (5 - \lambda_n) \right) (1 + \text{exp small})$$

We now convert the asymptotic formulae for $b_k^t$ into asymptotic formulae for $x_k(t)$ as $t \to \pm \infty$. We have

1. As $t \to +\infty$,

$$x_k - x_{k+1} = 2 \log t + 2 \log b_k$$

$$= 2t \left( x_{k+1} - x_k \right) + 2 \log \left( \frac{\sum_{i=1}^{k+1} \left( \lambda_i - \lambda_{k+1} \right)}{\sum_{i=1}^{k} \left( \lambda_i - \lambda_k \right)} \right) + o(1)$$

and as $t \to -\infty$,

$$x_k - x_{k+1} = 2t \left( x_{n-k} - x_{n-k+1} \right) + 2 \log \left( \frac{\sum_{i=n-k+1}^{n} \left( \lambda_i - \lambda_{n-k+1} \right)}{\sum_{i=n-k}^{n} \left( \lambda_i - \lambda_{n-k} \right)} \right) + o(1)$$

Now

$$\lim_{k \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k = -2 \sum_{k=1}^{n} a_k = -2 \sum_{i=1}^{n} \lambda_i = \text{const}$$

and so

$$\lim_{k \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k(t) = \sum_{k=1}^{n} x_k(0) - 2 \left( \sum_{i=1}^{n} \lambda_i \right) t$$

Summing (107.1), we obtain as $t \to +\infty$ for $1 \leq k \leq n-1$,
\[
(108.1) \quad x_k - x_N = 2 \sum_{j \geq 1} (\lambda_j - \lambda_k) + \frac{1}{2} \log \left( \frac{u_N(1)}{u_k(1)} \right) \prod_{\ell = 1}^{N-1} (\lambda_{\ell} - \lambda_N) + o(1)
\]

and no adding over \(k\)

\[
\sum_{k=1}^{N-1} x_k - (N-1) x_N = 2 \sum_{j \geq 1} (\lambda_j - \lambda_N) + \frac{1}{2} \log \left( \frac{u_N(1)}{u_k(1)} \right) \prod_{\ell = 1}^{N-1} (\lambda_{\ell} - \lambda_N) + o(1)
\]

In writing (107.3), we find that as \(t \to +\infty\)

\[
x_N(t) = \frac{1}{N} \sum_{k=1}^{N} x_k(t) = 2 t \lambda_N - \frac{1}{2} \log \left( \frac{u_N(1)}{u_k(1)} \right) \prod_{\ell = 1}^{N-1} (\lambda_{\ell} - \lambda_N) + o(1)
\]

which implies using (108.1) that as \(t \to +\infty\)

\[
(108.2) \quad x_k(t) = -2 t \lambda_k + \frac{1}{N} \sum_{k=1}^{N} x_k(0) - \frac{1}{2} \log \left( \frac{u_N(1)}{u_k(1)} \right) \prod_{\ell = 1}^{N-1} (\lambda_{\ell} - \lambda_N) + (\log 4) (N-k) + o(1)
\]

\[
-2 \sum_{k=1}^{N-1} \log \left( \frac{u_N(1)}{u_k(1)} \right) \prod_{\ell = 1}^{N-1} (\lambda_{\ell} - \lambda_N) + 2 \log \left( \frac{u_N(1)}{u_k(1)} \right) \prod_{\ell = 1}^{N-1} (\lambda_{\ell} - \lambda_N) + o(1)
\]
which reduces after some elementary algebra to

\[(109.1)\]
\[x_k(t) = -2t\lambda_k + \frac{1}{N} \sum_{j=1}^{N} x_j(t_0) - \frac{2}{N} \sum_{j=1}^{N} \log \left( \frac{u_k(t)}{u_j(t)} \right) \prod_{k=1}^{N} \left( \frac{2\lambda_k - 2\lambda_j}{e^\lambda_k - e^\lambda_j} \right) + o(1)\]

as \(t \to \pm \infty\), and a similar calculation using

\[(107.2)\]
\[\text{yields}\]

\[(109.2)\]
\[x_k(t) = -2t\lambda_{N-k+1} + \frac{1}{N} \sum_{j=1}^{N} x_j(t_0) - \frac{2}{N} \sum_{j=1}^{N} \log \left( \frac{u_{N-k+1}(t)}{u_{N-j+1}(t)} \right) \prod_{k=1}^{N} \left( \frac{2\lambda_{N-k+1} - 2\lambda_j}{e^{\lambda_{N-k+1}} - e^{\lambda_j}} \right) + o(1)\]

as \(t \to \pm \infty\). Summarizing, we have shown that for \(1 \leq k \leq N\),

\[(109.3)\]
\[x_k(t) = -2t\lambda_k + \beta_k^+ + o(1)\]

as \(t \to -\infty\)

\[(109.4)\]
\[x_k(t) = -2t\lambda_{N-k+1} + \beta_k^- + o(1)\]

as \(t \to +\infty\)

where

\[(109.5)\]
\[
\beta_k^+ = \frac{1}{N} \sum_{j=1}^{N} x_j(t_0) - \frac{2}{N} \sum_{j=1}^{N} \log \left( \frac{u_k(t)}{u_j(t)} \right) \prod_{k=1}^{N} \left( \frac{2\lambda_k - 2\lambda_j}{e^{\lambda_k} - e^{\lambda_j}} \right) \prod_{k=1}^{N} \left( \frac{2\lambda_k - 2\lambda_j}{e^{\lambda_k} - e^{\lambda_j}} \right)
\]

and

\[(109.6)\]
\[
\beta_k^- = \frac{1}{N} \sum_{j=1}^{N} x_j(t_0) - \frac{2}{N} \sum_{j=1}^{N} \log \left( \frac{u_{N-k+1}(t)}{u_{N-j+1}(t)} \right) \prod_{k=1}^{N} \left( \frac{2\lambda_{N-k+1} - 2\lambda_j}{e^{\lambda_{N-k+1}} - e^{\lambda_j}} \right) \prod_{k=1}^{N} \left( \frac{2\lambda_{N-k+1} - 2\lambda_j}{e^{\lambda_{N-k+1}} - e^{\lambda_j}} \right)
\]

We now compute the phase shift \(\beta_k - \beta_{N-k+1}\).
Consider \( x_{h(t+)} \sim -2 \lambda_h t \) as \( t \to +\infty \)
\( x_{N-h(t)} \sim -2 \lambda_h t \) as \( t \to -\infty \)

Thus particle \( x_{N-h(t)} \) travelling with velocity \( -2 \lambda_h t \) as \( t \to -\infty \), "transfers" its velocity to \( x_h(t) \) after collision as \( t \to +\infty \). This is why \( -\pi \) phase shift is defined as

\[
\beta_h^+ - \beta_{N-h-1}^- = \text{phase shift}
\]

\[
\beta_h^+ - \beta_{N-h-1}^- = \frac{1}{N} \sum_{j=1}^{N} \log \left( \frac{\sum_{\ell=1}^{k-1} \left( \frac{u_{h+j}(1)}{q_{ch}(1)} \right) \prod_{\ell=1}^{k-1} (2 \lambda_{h-j} - 2 \lambda_x) \prod_{\ell=1}^{j-1} (2 \lambda_{h-j} - 2 \lambda_j) \prod_{\ell=1}^{j-1} (2 \lambda_{h-j} - 2 \lambda_x)}{\prod_{\ell=1}^{k-1} (2 \lambda_{h-j} - 2 \lambda_{x+})} \right)
\]

We have

\[
\beta_h^+ - \beta_{N-h-1}^- = -\frac{1}{N} \sum_{j=1}^{N} \left[ \sum_{\ell=1}^{k-1} (2 \lambda_{h-j} - 2 \lambda_{x+})^2 \right] - \frac{1}{N} \sum_{j=1}^{N} \sum_{\ell=1}^{j-1} (2 \lambda_{h-j} - 2 \lambda_{x+})^2 + \frac{1}{N} \sum_{j=1}^{N} \sum_{\ell=1}^{j-1} (2 \lambda_{h-j} - 2 \lambda_j)^2
\]

Thus

\[
(110.1) \quad \beta_h^+ - \beta_{N-h-1}^- = \sum_{d=h}^{N} \phi_d \beta_k, \quad 1 \leq h \leq N
\]
where

\[
\Delta_{lh} = \log (2\lambda_i - 2\lambda_k)^2, \quad l \geq k
\]

\[
= -\log (2\lambda_i - 2\lambda_k)^2, \quad l < k.
\]

Here we have noted \( \sum_{l=N+1}^{N} \log (2\lambda_l - 2\lambda_k)^2 \equiv 0 \leq \sum_{l=1}^{N} \log (2\lambda_l - 2\lambda_i)^2 \)

and no

\[
\sum_{j=1}^{N} \sum_{k=j+1}^{N} \log (2\lambda_i - 2\lambda_k)^2 = \sum_{1 \leq j < k \leq N} \log (2\lambda_i - 2\lambda_k)^2
\]

\[
= \sum_{j=2}^{N} \sum_{i=1}^{j-1} \log (2\lambda_i - 2\lambda_j)^2
\]

\[
= \sum_{j=2}^{N} \sum_{i=1}^{j-1} \log (2\lambda_i - 2\lambda_j)^2
\]

\[
= \sum_{j=1}^{N} \sum_{k=1}^{j-1} \log (2\lambda_i - 2\lambda_k)^2.
\]

In particular, we see that for \( N = 2 \)

\[
\beta_i^+ - \beta_i^- = \phi_{2i} = \log (2\lambda_i - 2\lambda_2)^2
\]

\[
\beta_2^+ - \beta_2^- = \phi_{12} = -\log (2\lambda_1 - 2\lambda_2)^2.
\]

Thus the phase shifts

\[
\beta_k^+ - \beta_{N-k+1} = \sum_{i=k}^{N} \phi_{ih}.
\]

are just the same as if the interaction takes place two particles at a time.

Remark (111.2): This formula (10.1) for the phase shift is due to Moser.
(loc. cit.), which he derived using a very different argument.