(loc. cit), which he derived using a very different argument.

Lecture 7

Exercise 110.1

Use (100.1) and (107.5) to show that

$$x_i(t) = x_i(0) + \log \frac{Q_{i-1}(t)}{Q_{i+1}(t)},$$

where

$$x_i(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0) - \frac{2}{N} \sum_{i=1}^{N} x_i + \frac{(N-2)}{N} \log 2 \quad \frac{1}{N} \log Q_{i+1}(t)$$

$$= \frac{1}{N} \sum_{i=1}^{N} x_i(0) - \frac{2}{N} \sum_{i=1}^{N} x_i + \frac{(N-2)}{N} \log 2$$

$$+ \frac{1}{N} \log \left( \prod_{i=1}^{N} u_i^2(1) \right) V(\lambda) \frac{e^{2 \sum_{i=1}^{N} \epsilon_i}}{2 e^{2 \lambda_i}} u_i^2(1)$$

Use (112.2) (112.3) to deduce (109.3) and (109.4).

We now show that the Toda lattice is integrable

in the sense of Liouville i.e.

$$H_t(x, \lambda) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial \lambda_i} \right)^2 + \frac{1}{2} \sum_{i=1}^{N-1} e^{x_i - x_{i+1}}$$

has $N$ independent, Poisson commuting integrals $\{I_j\}$ on $\mathbb{C}^N$, $w = \sum_{i=1}^{N} x_i$
We take $I_\lambda = \lambda_i = \text{eigenvalues of } X_0$, $1 \leq i \leq n$.

We know these are integrals for the Toda flow; it remains to show that they are independent and Poisson commute.

**Theorem 113.1** Let $\lambda_i \neq \lambda_j$ be the eigenvalues of an $n \times n$ Jacobi matrix $X$. Then

\[(\lambda_k, \lambda_l) = 0, \quad 1 \leq k, l \leq n\]

**Proof:** Write

\[
X = \begin{pmatrix}
a_1 & b_1 & 0 \\
b_1 & a_2 & 0 \\
0 & \ddots & 0 \\
0 & \ddots & b_{n-1} \\
0 & \ddots & \ddots & a_n
\end{pmatrix}
\]

Then if $X(t), X(t+\epsilon) = X + \epsilon$ is any small perturbation of $X$, $X(t) \in J_n$, we have from $(X(t) - \lambda_j(t)) u_j(t) \epsilon = 0$,

\[
(X(t) - \dot{\lambda}_j(t)) u_j(t) \epsilon + (X(t) + \lambda_j(t)) \dot{u}_j(t) \epsilon = 0,
\]

\[
( u_j(t), \dot{X}(t) - \dot{\lambda}_j(t)) u_j(t) \epsilon ) = - (u_j(t), (X(t) + \lambda_j(t)) \dot{u}_j(t) \epsilon )
\]

\[
= - ( (X(t) u_j(t), \dot{u}_j(t) \epsilon ) - \lambda_j(t) u_j(t), \dot{u}_j(t) \epsilon )
\]
= -2 \lambda_i(0) \ (u_i(0), \dot{u}_i(0))

But as the eigenvalues of \(X(e)\) are simple, we can, by the results on perturbation theory noted earlier (see Lecture 5), assume that \(u_i(e)\) is smooth and normalize \(\langle u_i(0) \rangle = 1\), which implies \(\langle u_i(0), u_i(0) \rangle = 1\). Hence we conclude that

\[
\dot{u}_i(0) = (u_i(0), \dot{u}_i(0) u_i(0))
\]

Let \(E_{qp}\) denote the \(N 	imes N\) matrix with 1 in the \(qp\) position and zero elsewhere, \(1 \leq q, p \leq N\). That is,

\[
E_{qp}(i,j) = \delta_{iq} \delta_{jp} \quad (1 \leq i, j \leq N)
\]

From (114.1), we obtain, in particular

\[
\frac{\partial \lambda_i}{\partial a_p} = \langle u_i, E_{de} u_i \rangle = u_i^T(e) \quad (1 \leq e \leq N)
\]

and

\[
\frac{\partial \lambda_i}{\partial b_o} = \langle u_i, (E_{de} + E_{a_k}) u_i \rangle = 2 u_i^2(e) u_i(e) \quad (1 \leq e \leq N).
\]
Hence

\[ \frac{\partial \xi}{\partial y_i} = \frac{1}{2} \frac{\partial \xi}{\partial x_i} u_j^2(i) \quad \text{for } i \leq j \]

and

\[ \frac{\partial \xi}{\partial x_i} = \frac{\partial \xi}{\partial x_i} + \frac{\delta x_i}{\partial b_i} \frac{\partial b_i}{\partial x_i} + \frac{\delta x_i}{\partial b_{i-1}} \frac{\partial b_{i-1}}{\partial x_i} = \frac{\delta x_i}{\partial b_i} \frac{1}{2} b_i - \frac{\delta x_i}{\partial b_{i-1}} \frac{1}{2} b_{i-1} = u_j(i) u_j(i+1) b_i - u_j(i-1) u_j(i) b_{i-1} = u_j(i) b_i - u_j(i-1) b_{i-1}, \quad 1 \leq i \leq N \]

where

\[ b_0 = b_N = 0 \]

Claim: For \( 1 \leq i \leq N-1 \)

\[ (\xi - \lambda_k) \sum_{i \leq i} u_k(i) u_j(i) = b_i \left( u_k(m) u_j(m+1) - u_k(m+1) u_j(m) \right) \]

Indeed

\[ \lambda_j \sum_{i \leq i} u_k(i) u_j(i) = \sum_{i \leq i} u_k(i) \left( \lambda_j \right)_i = \sum_{i \leq i} u_k(i) \left( b_i u_j(i-1) + a_i u_j(i) + b_{i-1} u_j(i) \right) \]

\[ = \sum_{i \leq i} u_k(i-1) u_j(i) + \sum_{i \leq i} u_k(i) u_j(i) + \sum_{i \leq i} u_k(i) a_i u_j(i) + \sum_{i \leq i} u_k(i) b_i u_j(i) \]
\[
\sum_{i=1}^{m} \left( b_{i-1} u_k(i-1) + a_{i} u_k(i) + b_{i} u_{k+1}(i) \right) u_{j}(i) \\
+ b_{m} u_{k}(m) u_{j}(m+1) - b_{0} u_{k}(0) u_{j}(1) \\
- b_{m} u_{k}(m+1) u_{j}(m) + u_{k}(m) u_{j}(1) b_{0} \\
= b_{m} (u_{k}(m) u_{j}(m+1) - u_{k}(m+1) u_{j}(m))
\]

which proves the Claim. Note that (115.9) is just summing by parts.

Now from (115.1) (115.2)

\[
(116.1) \quad \left< \lambda_{k}, \lambda_{j} \right> = \sum_{i=1}^{N} \left( \frac{\partial \lambda_{k}}{\partial x_{i}} \frac{\partial \lambda_{j}}{\partial y_{i}} - \frac{\partial \lambda_{k}}{\partial y_{i}} \frac{\partial \lambda_{j}}{\partial x_{i}} \right)
\]

\[
= \sum_{i=1}^{N} \left[ a_{k}(i) (u_{k}(i+1) b_{i} - a_{k}(i-1) b_{i-1}) + \frac{1}{2} a_{k}(i) u_{j}(i) \left( u_{j}(i+1) b_{i} - a_{j}(i-1) b_{i-1} \right) \right. \\
- \left. \left( \frac{1}{2} a_{k}(i) u_{j}(i) \right) \left( u_{j}(i+1) b_{i} - a_{j}(i-1) b_{i-1} \right) \right]
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} a_{k}(i) u_{j}(i) \left[ b_{i} (u_{k}(i) u_{j}(i) - u_{k}(i-1) u_{j}(i)) + b_{i-1} (u_{k}(i) u_{j}(i) - u_{k}(i-1) u_{j}(i)) \right]
\]

Now from (115.3)
\[(\lambda_j - \lambda_k) \ u_k(i) \ u_j(i) = b_i \ (a_k(i+1) u_j(i+1) - a_k(i) u_j(i))
\]
\[- b_{i-1} \ (a_k(i-1) u_j(i) - u_k(i-1) u_j(i))
\]
\[= a_k(j(i) - a_k(j(i-1))
\]

Thus,
\[a_k^2(i) - a_k^2(i-1) = (a_k(j(i) - a_k(j(i-1)))^2
\]
\[= (\lambda_i - \lambda_k) u_k(i) u_j(i) (a_k(j(i) + a_k(j(i-1))
\]

It follows from (116.1) that
\[\begin{align*}
\sum_{k=1}^{N} a_k^2(i) - a_k^2(i-1) &= \frac{1}{2} \sum_{j=1}^{N} (a_k^2(N) - a_k^2(0)) \\
&= 0
\end{align*}
\]

As \( b_N = b_0 = 0 \).

This completes the proof of (113.2). \square

**Theorem 117.1**

Let \( \lambda_1, \ldots, \lambda_N \) be the eigenvalues of an \( N \times N \) Jacobi matrix \( X \). Then \( \lambda_1, \ldots, \lambda_N \) are independent.

**Proof:** We must show that if
\[(117.2)
\]
\[\sum_{k=1}^{N} \lambda_k = 0
\]
for some $\gamma_i, i = 1, \ldots, N$, then $\gamma_1 = \gamma_2 = \cdots = \gamma_N = 0$.

From (117.2), we have

\[ (118.1) \quad \delta = \sum_{k=1}^N \frac{\partial \gamma_k}{\partial x_0} = \sum_{k=1}^N \delta_k u_k(i) \left( u_k(i+1) b_{i+1} - u_k(i+1) b_{i+1} \right), \quad i \in \mathbb{Z} \cap M \]

and

\[ (118.2) \quad 0 = \sum_{k=1}^N \frac{\partial \gamma_k}{\partial y_i} = -\sum_{k=1}^N \delta_k u_k(i), \quad i = 1, \ldots, N \]

From (118.1), we obtain

\[ L_i = \sum_{k=1}^N \delta_k u_k(i+1) u_k(i) b_{i+1} = \sum_{k=1}^N \delta_k u_{k+1} b_k(i+1) b_{i+1} = L_{i+1} \]

But $L_0 = 0$ and no $L_i = 0$ for $1 \leq i \leq N$.

$L_N$ is trivially equal to zero as $b_N = 0$, but $b_i > 0$ for $1 \leq i < N$.

and no we conclude

\[ (118.3) \quad \sum_{k=1}^N \delta_k u_k(i+1) u_k(i) b_{i+1} = 0, \quad 1 \leq i \leq N-1. \]

From (118.2), we have

\[ (118.4) \quad \sum_{k=1}^N \delta_k u_k^2(i) = 0, \quad 1 \leq i \leq N. \]

Now for $1 \leq i \leq N$, we have from (118.3) and $\gamma_k = u_k$

\[ 0 = b_i \sum_{k=1}^N \delta_k u_k(i+1) u_k(i+1) \]
\[
(11.0) \quad \sum_{k=1}^{N} \alpha_k \lambda_k \Delta_k \mathbf{u}_k(i) \mathbf{u}_k(i+1) = \sum_{k=1}^{N} \beta_k \left( \lambda_k - \alpha_k \right) \mathbf{u}_k(i) \mathbf{u}_k(i+1)
\]

\[
= \sum_{k=1}^{N} \beta_k \left( \beta_i \mathbf{u}_k(i+1) + \beta_{i-1} \mathbf{u}_k(i-1) \right) \mathbf{u}_k(i+1)
\]

\[
= \beta_{i-1} \sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i-1) \mathbf{u}_k(i+1)
\]

On the other hand,

\[
\sum_{k=1}^{N} \alpha_k \lambda_k \mathbf{u}_k(i) \mathbf{u}_k(i+1) = \sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i) \left( \lambda_k - \alpha_k \Delta_k \right) \mathbf{u}_k(i+1)
\]

\[
= \sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i) \left( \beta_i \mathbf{u}_k(i+2) + \beta_i \mathbf{u}_k(i) \right)
\]

\[
= \beta_i \sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i) \mathbf{u}_k(i+2)
\]

Thus for \(1 \leq i \leq N-1\),

\[
b_{i+1} \sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i) \mathbf{u}_k(i+2) = \beta_{i-1} \sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i-1) \mathbf{u}_k(i+1)
\]

Now for \(i = 1\), the LHS = 0, and no as \(b_1 \neq 0\) we have from the RHS,

\[
\sum_{k=1}^{N} \alpha_k \mathbf{u}_k(i) \mathbf{u}_k(i+1) = 0
\]

Now for \(i = 2\), the RHS = 0, and as \(b_3 \neq 0\), we conclude that
\[ \sum_{k=1}^{N} \delta_k u_k(i) u_k(i+c) = 0 \quad (i = 1, \ldots, N-2) \]

Inserting this relation into (118.1), we obtain:

\[ \sum_{k=1}^{N} \delta_k \lambda_k u_k(i) u_k(i+1) = 0 \quad (1 \leq i \leq N-1) \]

Thus we see that (118.3), (118.4) imply (118.0) (120.2).

We can clearly repeat the above calculation and conclude that for \( 1 \leq i \leq N-1 \):

\[ \sum_{k=1}^{N-1} \lambda_k \delta_k u_k(i) u_k(i+c) = 0 \quad (i = 1, \ldots, N) \]

and

\[ \sum_{k=1}^{N-1} \lambda_k \delta_k u_k(i) u_k(i+1) = 0 \quad (1 \leq i \leq N-1) \]

But \( \lambda_k \delta_k \) is a Vandermonde matrix and so for any \( i \in \{1, \ldots, N\} \), we must have

\[ \delta_k u_k^2(i) = 0 \quad (k = 1, \ldots, N) \]

Also \( \sum_{i=1}^{N} u_k^2(i) = 1 \) and so we have \( \delta_k = 0 \), \( k = 1, \ldots, N \).