Before we begin studying the intrinsic properties of random matrices, a few more words on what it means to model a system by RMT. We mean the following: Suppose we are investigating some quantities \( \{a_k\} \) in a neighborhood of some point \( \Lambda \), say. The \( a_k \)'s are to be compared with the eigenvalues \( \{\lambda_k\} \), in a neighborhood of some point \( \Lambda \), of a matrix taken from some random matrix ensemble. If the statistics of the \( a_k \)'s, appropriately centered and scaled,

\[
a_k \quad \rightarrow \quad \tilde{a}_k = (a_k - \Lambda) \theta_k
\]

some appropriate scaling factor

are described by the statistics of the \( \lambda_k \)'s, appropriately centered and scaled
\[ \lambda_n \to \tilde{\lambda}_n = (\lambda_n - \Lambda) \gamma \]

some appropriate scaling factor

Then we say that the \( \tilde{\lambda}_n \)'s are modeled by

**random matrix theory**

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**Intrinsic Theory: RMT**

We will restrict our analysis to ensembles of \( N \times N \) Hermitian matrices

\[ \tilde{M} = (\tilde{M}_{ij})^2, \quad \tilde{M} = \tilde{M}^* \]

There are many other ensembles which are of interest, for example ensembles of unitary matrices such as the Circular Unitary Ensemble (CUE) endowed with Haar measure \( K \). Or COE, the ensemble of orthogonal matrices, also endowed
with Haar measure, now on the orthogonal group.

Or sample covariance matrices of the form

\[ M = XX^* \]

where \( X \) is an \( NxP \) rectangular matrix,

\( \begin{pmatrix} \text{columns} \ x_1, \ldots, x_p \end{pmatrix} \)

whose columns \( x_1, \ldots, x_p \) are identically

\( \text{Gaussian} \)

distributed \( \text{samples with mean } \mu \) and covariance \( \Sigma \). Each of these ensembles are useful in various applications (sample covariance matrices are at the heart of what is called Principal Component Analysis, a key tool in statistics and mathematical finance). Although we will not analyze these ensembles (see the various recommended texts and the ref's key content), the analysis of Hermitian ensembles provides a
model and a guide for all random matrix ensembles.

We follow refs (2) and (3) for much of what follows. There are three kinds of Hermitian matrix ensembles which are of interest (Dyson's "three-fold way", see [Mehta]): These consist of

\[(22.1)\] \(N \times N\) Hermitian matrices \(M = (M_{ij}) = M^\dagger\)

\[(22.2)\] \(N \times N\) real Hermitian (i.e. real symmetric) matrices \(N = (M_{ij}) = M = M^T\)

\[(22.3)\] \(2N \times 2N\) Hermitian self-dual matrices \(M = (M_{ij}) = M^\dagger = J M^T J^T\)

where \(J = \text{diag}(\sigma, \sigma, \ldots, \sigma)\), \(\sigma = (0, 1)\).

For all three classes of ensembles, the probability distribution on the matrices is given by...
\[(23.1)\quad P_n(M)\,dM = \frac{1}{Z_n} F(M)\,dM\]

where \(dM\) is Lebesgue measure on the algebraically independent entries of \(M\), \(F(M)\) is a
convergence factor to ensure that \(P_n(M)\,dM\)
is a (finite) probability measure, and
\[Z_n = \int F(M)\,dM\]
is the normalization factor (sometimes called the
partition function). (Discrete measures on the
matrices, e.g., \(M_{ij} = 1\) or \(-1\) with equal probability,
are also of interest, but we will not consider
them.)

Now \(N \times N\) Hermitian matrices \(M_{jk} = M_{jk}^* + iM_{jk}\)
\[= \overline{M}_{kj}\]
depend on \(N + 2\cdot N(N-1)/2 = N^2\).
and

\[ (24.1) \quad dM = \prod_{k=1}^{N} dM_{kj} \prod_{1 \leq k \leq j \leq N} dM_{kj} \prod_{1 \leq k < j \leq N} dM^{x}_{kj} \]

For (real) symmetric \( N \times N \) matrices \( M_{ki} = M_{ik} \), the matrices depend on \( N + N(N-1) - \frac{N(N+1)}{2} \) real variables and

\[ (24.2) \quad dM = \prod_{1 \leq k \leq j \leq N} dM_{kj} \]

For \( 2N \times 2N \) Hermitian self dual matrices \( M \), write \( M \) in the form of \( 2 \times 2 \) blocks \( M = (m_{ij}) \), then (exercise: see also (3)) the condition \( M = M^{\ast} \odot T' \) implies. Thus

\[ (24.3) \quad m_{kj} = m_{jk}^{\ast} \quad (1 \leq j, k \leq N) \]

and the condition \( M^{\ast} = -J M^{T} J \) implies

\[ (24.4) \quad m_{ij} = -\sigma m_{ij} \quad (1 \leq j, k \leq N) \]
From (24.4) we learn that for

\[ (25.1) \]

\[ m_{ij} = d_{ij} I + \beta_{ij} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \delta_{ij} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

where \( d_{ij}, \beta_{ij}, \delta_{ij} \) and \( \delta_{ij} \) are real.

For \( k > j \), \( m_{kj} = m_{kj}^T \) by (24.3).

And for \( k = j \), by (25.1), (24.3), \( m_{kk} = d_{kk} I \), \( d_{kk} \) real. Relation (25.1) shows that the \( m_{ij} \)'s are real quaternions. Thus Hermitian self-dual matrices in \( \mathbb{H}^{2 \times 2} \) have a block structure

\[ M = \begin{pmatrix} m_{ij} & 0 \\ 0 & m_{ij}^T \end{pmatrix} \]

and hence depend on

\[ N + 4 \frac{N(N-1)}{2} = 2N^2 - N \]

real variables.
Hence

\[ (26.1) \quad d^N r = \prod_{k=1}^{N} dx_k \prod_{k \neq j} d^2 \phi_k d^2 \phi_j \]

Historically, two kinds of Hermitean ensembles are of interest:

Those that are invariant under a natural conjugation of the matrices, and those where the algebraically independent entries of the matrices are also statistically independent. The first class of ensembles are called invariant ensembles; the second class are called Wigner ensembles.

We first consider invariant ensembles.

Wigner as a subject goes back to the work of statisticians in the 1920s, but the subject was introduced into physics only in the 1950s.
by Wigner who was interested in the analysis of scattering resonances in neutron scattering theory (the first bookend mentioned above). Such resonances reflect the precise properties of the Hamiltonian $H$ which describes the neutron-heavy atom system, but the degrees of freedom are so large that one could not hope to solve the system for the resonances even numerically. What was needed was a model for the resonances and, taking into account various experimental observations, Wigner was led to posit even random matrices as a model for $H$. Now, a matrix $M$, say, has no intrinsic meaning: if one changes the basis, the
matrix changes by conjugation \( M \to M' = U M U^* \).

For this reason Wigner singled out ensembles that were invariant under conjugation by (appropriate) classes of matrices. These are what we called to invariant ensembles. Wigner ensembles, where the entries are independent, are appropriate for statistical/data-type problems, for example, where the matrices have intrinsic meaning (for more info see [Meh] [G])

Invariance for the Hermitian ensembles

\((22.1) \quad (22.2) \quad (22.3)\) means the following:

For \((22.1)\), \( P_n(m) \) must be invariant under the conjugation \( M \to U M U^* \) for all unitary matrices \( U \).

For \((22.2)\), \( P_n(m) \) must be invariant under conjugation \( M \to U M U^T \) for all (real) orthogonal
matrices $U$

For (22.3) $P_n(m)$ and must be invariant under the conjugation $M = U M U^*$ for all unitary and symplectic matrices $U$, $U U^* = I$ and $U J U^T = J$. Note that if $M$ is a self-dual Hermitian matrix, then so is $M' = U M U^*$ for any unitary/symplectic matrix $U$. Indeed, $M'$ is clearly Hermitian, so we only need to show that $M'^* = J M'^T J^T$

But this is a simple exercise.

We first consider $N \times N$ Hermitian matrices.

For any Hermitian matrix $M$ let

$$F_1 = (M_{11}, \ldots, M_{nn}, M_{12}^{R}, M_{12}^{I}, M_{21}^{R}, M_{21}^{I}, M_{13}^{R}, M_{13}^{I}, \ldots, M_{n-1,n}^{R}, M_{n-1,n}^{I}) \in \mathbb{R}^{N^2}$$
In order to show that

\[ (30.0) \quad \text{det} \frac{\partial \mathbf{M}}{\partial \mathbf{M}'} = 1 \]

where \( \mathbf{M}' = \mathbf{U} \mathbf{K} \mathbf{U}' \), we must show that

\[ \text{det} \frac{\partial \mathbf{K}}{\partial \mathbf{K}'} = 1. \]

But clearly \( \mathbf{T} \mathbf{M}^2 = \mathbf{T} \mathbf{M}'^2 \)

\[ \Rightarrow \sum \mathbf{M}_{jk} \mathbf{M}_{kj} = \sum \mathbf{M}'_{jk} \mathbf{M}'_{kj} \quad \Rightarrow \]

\[ \sum_{j} \mathbf{M}_{jj}^2 + 2 \sum_{j<k} |\mathbf{M}_{jk}|^2 = \sum_{j} \mathbf{M}'_{jj}^2 + 2 \sum_{j<k} |\mathbf{M}'_{jk}|^2 \]

or

\[ (30.1) \quad \sum_{j} \mathbf{M}_{jj}^2 + 2 \sum_{j<k} (\mathbf{M}_{jk}^a)^2 + 2 \sum_{j<k} (\mathbf{M}_{jk}^b)^2 \]

\[ = \sum_{j} \mathbf{M}_{jj}^2 + 2 \sum_{j<k} (\mathbf{M}'_{jk}^a)^2 + 2 \sum_{j<k} (\mathbf{M}'_{jk}^b)^2 \]

In other words, if \( \mathbf{D} \) is the \( N^2 \times N^2 \) diagonal matrix

\[ \mathbf{D} = \text{diag} (1, \ldots, 1, 2, \ldots, 2) \quad (N^2 \text{ 1's}) \]

then (30.1) shows

\[ (30.2) \quad (\mathbf{M}, \mathbf{D} \mathbf{M}) = (\mathbf{M}', \mathbf{D} \mathbf{M}') \]

Thus if we write \( \mathbf{M}' = \mathbf{T} \mathbf{M} \) for some \( N^4 \times N^4 \)
matrix $T$, then (30.2) shows that $T$ is orthogonal with respect to the inner product induced by $D$ on $\mathbb{R}^n$. i.e.,

$$T^TD T = D$$

Thus, $$(\det T)^2 = \det T^T \det T = 1$$ and hence

$$\det \left( \frac{\partial \tilde{\mu}'}{\partial \tilde{\mu}} \right) = \det T = 1$$ as desired.

We conclude that $p_n(m) \det T$ is invariant if and only if

$$F(m) = \int F(m) \, dm = \frac{F(m')}{{\int F(m') \, dm'}}$$

for all unitary $U$, $m' = U m U^*$. And all wrapping $\mu$. Setting $\mu = 0$ in (31.1) we see that in fact

$$F(m) = F(m') = F(U m U^*)$$

We will always assume in these lectures.
that \( F(M) \) is of the form

\[
F(M) = \exp(-iQ(M))
\]

where \( Q(x) \) is a real valued function on \( \mathbb{R} \) which grows sufficiently rapidly as \( |x| \to \infty \) so that

\[
\int e^{-iQ(x)} \, dx < \infty.
\]

Here \( Q(M) \) is defined by the spectral calculus

for \( M \) if \( M = U \Lambda U^* \), \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \)

in the spectral representation for \( M \), \( U \) unitary.

Then \( Q(M) = U Q(\Lambda) U^* = U \left( \begin{array}{ccc} Q(\lambda_1) & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & \cdots & Q(\lambda_n) \end{array} \right) U^* \)

For example, if \( Q(x) = x^2 \), then

\[
F(M) = \exp(-iM^2)
\]

Clearly \( F(M) = F(UMU^*) \)

This gives rise to the \textbf{Gaussian Unitary Ensemble} (GUE) mentioned in the first lecture,

\[
P_n(M) \, dM = \frac{1}{Z_n} e^{-\frac{1}{2} M^2} \, dM.
\]
The ensembles of Hermitian matrices with prob. distribution

\[
P_n(t) \, dx = \frac{1}{Z_n} e^{-\frac{1}{2} Q(t_n)} \, dx
\]

are called the unitary ensembles (UE's).

Which \( Q \) is the "right" \( Q \) to choose? The remarkable fact is that, on the right scale, it does not matter. In other words, on the right scale as \( n \to \infty \), one has precisely the same fluctuation statistics for the eigenvalues, independent of the particular choice one makes for \( Q \). This phenomenon is known as universality: proving universality is one of the chief tasks in the intrinsic theory of RMT.

For \( Q \), we may choose, for example,
any function \( Q(x) \) such that

\[(34.1) \quad Q(x) \geq c x^k + d \]

for some constants \( c, d \). For then

\[ -pA(M) \geq c \, tr \, M^2 + nd \text{ as by (30.1)} \]

\[ e^{-pA(M)} \, dM = e^{-Nd} \, e^{-kM^2} \, dM \]

\[ = e^{-Nd} \, e^{-M_{11}^2} \, dM_{11} \ldots e^{-M_{NN}^2} \, dM_{NN} \]

\[ e^{-2(M_{11}^2 \, dM_{11})} \ldots e^{-2(M_{NN}^2 \, dM_{NN})} \]

and the RHS clearly has finite integral. In fact

\[ e^{-pA(M)} \, dM \text{ clearly has finite moments} \]

\[(34.2) \quad \int |M_{ij}|^k \, e^{-pA(M)} \, dM < \infty \text{ for all } i,j, \]

\[ \text{and all } k \geq 0. \]

In particular, we can take any polynomial

\[(34.3) \quad Q(x) = \alpha x^2 + \ldots \quad \alpha > 0. \]

As we will see, any \( Q(x) \) at
\[
\frac{\omega_1}{\log q} \rightarrow +\infty \quad \text{as} \quad 1+1 \rightarrow +\infty
\]

\[\text{gives rise to an ensemble with finite moments.}\]

\[\text{We leave it as an exercise to show}\]

\[\text{that again invariance under orthogonal conjugation,}\]
\[\text{and under unitary/symplectic conjugation, for (22.2) and (22.3) respectively, requires the prob. dist.}\]
\[\text{to have the form}\]
\[
P_n(x) dx = \frac{1}{Z_n} F(x) dx
\]
\[\text{and again we always assume that } F(x) \text{ is of}\]
\[\text{the form } -\text{tr} \mathcal{Q}(x) \text{ for } \mathcal{Q} \text{ as above; thus}\]
\[
P_n(x) dx = \frac{e^{-\text{tr} \mathcal{Q}(x)}}{Z_n} dx
\]

One calls such ensembles **Orthogonal Ensembles (OE's)**/**Symplectic Ensembles (SE's)**.

If we choose \(\mathbf{G} \times 1 = x^T\) for (22.2) we obtain the Gaussian **Orthogonal Ensemble (GOE)**.

And for (22.3) we obtain the Gaussian **Symplectic Ensemble (GSE)**. Note that the only invaraint
ensembles that are also Wigner ensembles are

GUE, GOE and GSE (why?).

In order to compute key statistics, such

as the probability that there are no eigenvalues

in a gap \((a, b)\) (the so-called gap probability),

or the \(n\)-point correlation function, etc., it is

useful to use the spectral theorem for \(M\),

\[
M = U \Lambda U^* \]

as a change of variables

\[
M \mapsto (\Lambda, U) \]

\[
dM = \frac{1}{2 \det(\Lambda, U)} \, d\Lambda \wedge du \]

It is instructive to consider first the case

where \(M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\) is a real symmetric matrix
Then
\[ M = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^T \]

where
\[ U = \begin{pmatrix} \cos \theta & -i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}, \quad 0 < \theta < 2\pi. \]

Then
\[
\begin{align*}
  a &= \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta \\
  b &= (\lambda_1 - \lambda_2) \cos \theta \sin \theta \\
  c &= \lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta
\end{align*}
\]

and hence
\[
\frac{\partial (a, b, c)}{\partial (\lambda_1, \lambda_2, \theta)} = \begin{pmatrix}
  \cos^2 \theta & \sin^2 \theta & -2(\lambda_1 - \lambda_2) \cos \theta \sin \theta \\
  \sin \theta \cos \theta & -\sin \theta \cos \theta & (\lambda_1 - \lambda_2) (\cos^2 \theta - \sin^2 \theta) \\
  \sin^2 \theta & \cos^2 \theta & \lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta
\end{pmatrix}
\]

Thus
\[
(37.1) \quad \left| \det \left( \frac{\partial (a, b, c)}{\partial (\lambda_1, \lambda_2, \theta)} \right) \right| = |\lambda_1 - \lambda_2| f_1(\theta)
\]

where
\[
f_1(\theta) = \left| \det \begin{pmatrix}
  \cos^2 \theta & \sin^2 \theta & -2\lambda_1 \sin \theta \\
  \frac{1}{2} \sin \theta & -\cos \theta & 2\lambda_1 \cos \theta \\
  \sin^2 \theta & \cos^2 \theta & \lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta
\end{pmatrix} \right| = 1 > 0
\]

If \( M \) is a \( 2 \times 2 \) Hermitian, we find similarly (example)
\[
(37.2) \quad \left| \det \frac{\partial (M)}{\partial (\lambda, u)} \right| = (\lambda_1 - \lambda_2) f_2(\lambda, u), \quad f_2(\lambda) > 0
\]
and if $M$ is a $4 \times 4$ Hermitian self-dual matrix

\[(38.1) \quad \left| \det \frac{\partial^2}{\partial (n, u)} \right| = (4 - 2 \beta) \quad f_4(u), \quad f_4(u) > 0\]

Thus for invariant ensembles, we see (at least for these low-dimensional examples) the eigenvalues and eigenvectors are statistically independent.

\[(38.2) \quad e^{-\beta \Omega(u)} \prod_{i<j} (\lambda_i - \lambda_j)^{1/\beta} \prod_{p} f_p(u) du\]

where

\[(38.3) \quad \begin{cases} \beta = 1 & \text{for orthogonal ensembles} \\ \beta = 2 & \text{for unitary ensembles} \\ \beta = \infty & \text{for symplectic ensembles} \end{cases}\]

In the well-known analogy between\(\beta = T\) and statistical mechanics (see [Reference]) \(\beta\) corresponds to an inverse temperature.

For \(\beta = \infty\) for general $N$.

We will prove (38.2) (\(\beta = 2\) for orthogonal ensembles) and leave the case \(\beta = 1\) and \(\beta = \infty\) as exercises.
So suppose \( M = \mathbf{F} \mathbf{F}^T \) is an \( n \times n \) symmetric matrix. Then \( M \) has a spectral decomposition

\[
M = U \Lambda U^T
\]

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) and \( U \) is orthogonal.

However, the map

\[
M \mapsto (\Lambda, U)
\]

is not well-defined for all \( M \). Indeed if \( M = I \), then any \( U \) will do. However, if

the eigenvalues of \( M \) are simple, \( \lambda_i \neq \lambda_j \) for \( i \neq j \),

then the columns \( u_1, \ldots, u_n \), which are the normalized eigen vectors of \( U \), \( M u_j = \lambda_j u_j \), are defined up to multiplication by \( \pm 1 \).
This means that the map
\[ \tau: \mathbb{R}^n \rightarrow (\Lambda, \Omega) \]
is well-defined as a map from the set of matrices with simple spectrum into
\[ \{ \lambda_1 < \cdots < \lambda_n \} \times \text{O}(N)/\text{H}(N) \]
is well-defined and we can compute its Jacobian.

This is the approach followed in (2) (3). Note that a critical element in this approach is to show that the set of symmetric matrices with simple spectrum has full measure in its complement has measure 0. This is necessary to conclude that \( \tau : \mathbb{R}^n \rightarrow (\Lambda, \Omega) \) is a valid change of variables for integration. However, we will use
a slightly different approach suggested by Oliver Conway, which avoids calculations for the homogeneous space \( O(N)/H(N) \).

The key fact is the following.