$0 < \mu'(D_{1/4}(x_1 < 0)) \leq 0.$ It follows that

\begin{equation}
\text{supp } \mu' \subseteq [-1/2, 1/2]
\end{equation}

Note that since $V(x) > V_0 > -\infty$, and $\mu'$ has compact support,

$-\infty < V_0 \leq \int V(x) \mu'(x) \leq \infty$

and hence

\begin{equation}
\infty > E^\mu - V_0 > \log \log (1 - s^{1/2}) \mu'(1/q'(s)) > -\infty
\end{equation}

We say that $\mu'$ has finite logarithmic and potential energy.

We now consider uniqueness. Recall that a finite signed measure $\mu$ has a (unique) Jordan decomposition $\mu = \mu_+ - \mu_-$, where $\mu_+$ and $\mu_-$ are positive finite measures,

$0 < \mu_+ < \infty$, and are mutually singular and supported on disjoint sets. If $\mu_+$ and $\mu_-$ are $2$ finite measures,
$m = \mu_1 - \mu_2$ is a finite signed measure. We say that a real or complex valued function $f$ is integrable with respect to $|\mu| = \mu_1 + \mu_2$, and $m$

In this case

$$\int f \, d\mu = \int f \, d\mu^+ - \int f \, d\mu^-.$$

If $m = \mu_1 - \mu_2$ is any decomposition of the finite signed measure $\mu$ as a difference of finite measures, then it is easy to see that

$$\int f \, d\mu = \int f \, d\mu^+ - \int f \, d\mu^-$$

for all $f$ that are integrable with respect to $\mu$ and $\mu_1$ (and hence integrable with $\mu_2$).

**Lemma 12.2.1**

Let $\mu$ be a finite signed measure on $\mathbb{R}$ with mean zero if $\int f \, d\mu = 0$ and with compact support. Then

$$\int \left|\frac{1}{2} (f(x+y) + f(x-y)) - f(x)\right| \, d\mu(x) \leq 0.$$
Remark:

Inequality (122.2) needs some interpretation.

Although the function \( \log |x - y|^{-1} \) is bounded below on the compact support of \( \mu_1(x) \otimes \mu_2(y) \), the function is not bounded above. Therefore we do not know whether it is integrable with

\[
\mu_1(x) \otimes \mu_2(y) = \left( \mu_1(x) \otimes \mu_1(y) + \mu_2(x) \otimes \mu_2(y) \right)
- \left( \mu_1(x) \otimes \mu_2(y) + \mu_2(x) \otimes \mu_1(y) \right)
\]

where \( \mu_1 = \mu_1 - \mu_2 \) is any decomposable \( \mu_1 \) as a difference of measures.

So what we mean by (122.2) is that for any decompositions \( \mu_1 = \mu_1 - \mu_2 \), where \( \mu_1, \mu_2 \) are measures with compact support,

\[
\int \log |x - y|^{-1} \left( \mu_1(x) \mu_2(y) + \mu_2(x) \mu_1(y) \right)
\]

\[
> \int \log |x - y|^{-1} \left( \mu_1(x) \mu_2(y) + \mu_2(x) \mu_1(y) \right)
\]

\[
= 2 \int \log |x - y|^{-1} \mu_1(x) \mu_2(y)
\]

\[
= 2 \int \log |x - y|^{-1} \mu_2(x) \mu_1(y)
\]

Here there is no ambiguity since \( \log |x - y|^{-1} \) is bounded below on the compact set \( \text{supp } \mu \times \text{supp } \mu \) and hence both sides of
The inequality in (12.1) are well defined.

Of course \( \log|x-y| \) is integr. w.r.t both the measures \( \mu_1(x) \mu_1(y) \) and \( \mu_2(x) \mu_2(y) \). Then it is integrable w.r.t both the measures \( \mu_1(x) \mu_1(y) \) and \( \mu_2(x) \mu_2(y) \).

In this case \( |x-y| \) is integr. with respect to \( \mu_1(x) \otimes \mu_2(y) \) and so (12.2) is true as it stands.

Proof of Lemma 12.1: For any real \( \varepsilon > 0 \)

\[
\log(e^{\varepsilon} + e^{\varepsilon}) = \log e^{\varepsilon} + \int_0^\infty \frac{2t}{e^{t+\varepsilon}} dt
\]

\[
= \log e^{\varepsilon} + 2 \Im \int_0^\infty \frac{i}{t + i\varepsilon} dt
\]

\[
= \log e^{\varepsilon} + 2 \Im \int_0^\infty \frac{1}{-i} e^{i(t+i\varepsilon)} du
\]

\[
= \log e^{\varepsilon} + 2 \Im \int_0^\infty du e^{-2\mu} \frac{e^{i\mu}}{i\mu}
\]

\[
\Rightarrow \log e^{\varepsilon} + 2 \Im \int_0^\infty du e^{-2\mu} \frac{e^{i\mu} - 1}{i\mu}
\]

Hence for the function \( \log(e^{x-y} + e^{x+y}) \), which is
clearly integrable with respect to the (compactly supported measure) \( \mu \otimes \mu \),

\[
\int \log((x-y)^2 + \varepsilon^2) \, d\mu(x) \, d\mu(y)
\]

\[
= \int \log z^2 \, d\mu(x) \, d\mu(y) + \text{Im} \int_0^\infty du \, e^{-u} \left[ \int \mu(u) \, e^{i(x-y)u} \right]
\]

\[
= 2 \text{Im} \int_0^\infty du \, e^{-u} \frac{\hat{\mu}(u)}{iu} \quad (\text{as } \int \mu = 0)
\]

\[
= -2 \int_0^\infty du \, e^{-u} \frac{\hat{\mu}(u)}{u}
\]

Thus,

\[
(12.5.1) \quad -2 \int \log((x-y)^2 + \varepsilon^2) \, d\mu(x) \, d\mu(y) = 2 \int_0^\infty du \, e^{-u} \frac{\hat{\mu}(u)}{u}
\]

Note that \( \hat{\mu}(0) = \int d\mu(x) = 0 \), and as \( \hat{\mu}(u) \) is analytic in \( u \), there is no singularity in the integral on the RHS as \( u \to 0 \). Writing out (12.5.1) we find for any decomposition \( \mu = \mu_1 - \mu_2 \), where \( \mu_1, \mu_2 \) are measures with compact support

\[
\int \log((x-y)^2 + \varepsilon^2) \left( d\mu_1(x) \, d\mu_1(y) + d\mu_2(x) \, d\mu_1(y) \right)
\]

\[
= \int \log((x-y)^2 + \varepsilon^2) \left( d\mu_1(x) \, d\mu_1(y) + d\mu_2(x) \, d\mu_1(y) \right)
\]

\[
+ \int_0^\infty du \, e^{-u} \frac{\hat{\mu}_1(u)}{u}
\]
Letting $\varepsilon \to 0$, by the monotone convergence theorem (126)

we use here that $\log \left( (x-y)^{1+\varepsilon} \right)$ is bounded below
uniformly in $\varepsilon < 1$ for the measures have compact support.

\[
(126.1) \quad \int \log |x-y|^{-1} \left( \nu_1(x) \mu_2(y) + \nu_2(x) \mu_1(y) \right) dx dy = \int \log |x-y|^{-1} \left( \nu_1(x) \mu_2(y) + \mu_1(x) \nu_2(y) \right) dx dy
\]

\[
+ \int_0^\infty \frac{\mu_1(x)}{u} \, du
\]

which implies, in particular, inequality (122.2)

Now suppose $\mu, \tilde{\mu} \in \mathcal{M}_1(\mathbb{R})$ are two prob. dists.

for which $E^\mu = H(\mu) = H(\tilde{\mu}) = \inf_{\mu' \in \mathcal{M}_1(\mathbb{R})} H(\mu')$.

By the argument above, $\mu$ and $\tilde{\mu}$ have compact support.

$-\infty < E^\mu < \infty$, $\log |x-y|^{-1}$ is integrable with respect to $\mu \otimes \mu$ and $\tilde{\mu} \otimes \tilde{\mu}$. Hence it follows from (123.1) that $\log |x-y|^{-1}$ is integrable with respect to $\mu \otimes \mu$ and $\tilde{\mu} \otimes \tilde{\mu}$; all we need to
It is that \( \int x \cdot \delta^t - q \cdot \tilde{\mu} = 1 - 1 = 0 \). It follows that \( \log (x - y_1) \) is integrable with respect to the measure \( \nu \). 

Furthermore:

\[
\begin{align*}
\tilde{\mu} &= \mu + t (\tilde{\mu} - \mu) \quad \text{for } t \in \mathbb{R}.
\end{align*}
\]

\( f(t) = \int \log |x - y_1| \cdot \mu(\partial_k \Delta_k(y)) + \int \log |x - y_1| \cdot \mu(\partial_k \Delta_k(y)) + \int \log |x - y_1| \cdot \mu(\partial_k \Delta_k(y)) + \int \log |x - y_1| \cdot \mu(\partial_k \Delta_k(y))
\]

(Note by the above discussion all the integrals are well-defined and finite.) Now \( \tilde{\mu} - \mu \) has mean zero and compact support. Hence by (12.3), \( f(t) \) is a convex function of \( t \), and so, for \( 0 < t < 1 \),

\[
\begin{align*}
&\int f(t) - t f(0) + (1 - t) f(1) \leq (1 - t) \cdot |t| + t |f(1) + (1 - t) f(0)| \leq \frac{1}{2} |(1 - t) f(0) + t f(1)|.
\end{align*}
\]
\[ E^y = H(\mu_l - H(\tilde{\mu})) = E^y. \]

Thus, \( H(\mu^+) = E^y = \text{const.} \) In particular, by (126.1),

\[ 0 = \frac{1}{2} \tilde{f}''(0) = \int \log(x - y) \tilde{d}(\tilde{\mu} - \mu)(x) \delta(x - \tilde{y})(y) \]

\[ = \int \int \frac{\tilde{\mu} - \mu}{u} \frac{1}{u^2} \frac{1}{\mu} \]

This is \( \tilde{\mu}(u) = \tilde{\mu}(u) \forall u > 0 \), and hence \( \forall u \)

as \( \tilde{\mu}(u) = \tilde{\mu}(-u) \) as \( \tilde{\mu}(u) = \tilde{\mu}(-u) \)

Alternatively, \( \tilde{\mu}(u) \) and \( \tilde{\mu}(u) \) are entire. Thus

\( \tilde{\mu} = \mu \) as desired. This completes the proof of Thm 113.3.

Remark: The proof of Thm 113.3 follows the "standard"

path for the solution of a convex minimization

problem.

Recall the definition (128.1) of the correlation function \( R_i(x_1) = \)

(128.1) \[ R_i(x_1) = n \int P_n(x_1, x_2, \ldots, x_n) dx_2 \ldots dx_n \]
\( P_N(x_1, x_2, \ldots, x_N) = \frac{1}{Z_N} e^{-N \sum_{i=1}^{N} V(x_i)} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \, d^N x \)

and \( Z_N \) is the normalization factor (partition function).

\( Z_N = \int e^{-N \sum_{i=1}^{N} V(x_i)} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \, d^N x \).

Our goal is, eventually, to prove the following result:

**Theorem 12.9.3** Let \( V(x) \) be a continuous function satisfying (12.1):

\[ \frac{V(x)}{\log(x^2+1)} \to \infty \text{ as } x \to \infty. \]

Then

\[ \frac{1}{N} \int R_N(x) \, dx \to \text{qued} \]

as \( N \to \infty \).

By (95.2),

\[ \int R_N(x) \, dx = \text{Exp} \left( \text{# eigenvalues in } \Sigma \right), \]

where (12.9.4) says that

\[ \text{Exp} \left( \text{# eigenvalues in } \Sigma \right) \to \mu_d(\Sigma). \]

In other words, qued is the density of states.
Recall from (13.1), \( k(t,s) = \log |t-s|^{-1} + \frac{1}{2} V(t) + \frac{1}{2} V(s) \).

(130.1) Let \( K_N(x) = \sum_{i=1}^{N} k(x_i, x_j) \).

\[
= \sum_{i \leq j} \log |x_i - x_j|^{-1} + \frac{(N-1)}{2} V(x_i)
\]

The key to the proof of (129.4) is following (large deviation) estimate.

**Lemma 130.2**

Let

\[
P_N(x) \, dx = \frac{1}{2^N} e^{-N \sum_{i=1}^{N} V(x_i)} \prod_{i < j} (x_i - x_j) \, dx
\]

be a prob. meas. (as in (129.1) above) and let \( \eta > 0 \) be given. Set

(130.3) \( A_{N, \eta} = \{ x \in \mathbb{R}^N : \frac{1}{N} K_N(x) \leq E^N + \eta \} \)

and let \( a > 0 \) be any positive number. Then \( \mathbb{F}^* N^* = N^*(m) \)

which depends on \( \eta \), but not on \( a \), \( x \), so

(130.4) \( P_N(\mathbb{R}^N \setminus A_{N, \eta + a}) \leq e^{-aN^2}, \quad N \geq N^*(\eta) \)

**Proof:** For any \( \varepsilon > 0 \), set

(130.5) \( \psi_{\varepsilon}(x) = \frac{1}{2 \varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \mu \, dt \)
where \( q \) is the eqm. meas. for \( V \), \( H \)

\[ H(\mu^0) \leq \infty \], \( \mu^0 \) has no point masses (why?).

and hence \( \varphi_\varepsilon(x) \to 0 \) is a continuous function of \( x \). In addition, \( \varphi_\varepsilon \) has compact support, so does \( \varphi_\varepsilon(x) \). Also \( \varphi_\varepsilon(x) \) is the convolution of \( \mu^0 \) with a function of mean 1 and no \( \int \varphi_\varepsilon(x) dx = 1 \). An elementary argument shows that \( \varphi_\varepsilon(x) dx \to \mu^0 \) as \( \varepsilon \to 0 \).

We want to show more, viz., as \( \varepsilon \to 0 \)

\[ A(\varphi_\varepsilon) - H(\mu^0) = \varepsilon^V \]  

Interchanging the order of integration, and then rescaling, we obtain

\[ \iint \log |t-s| \varphi_\varepsilon(t) \varphi_\varepsilon(s) \, dt \, ds \]

\[ = \iint \varphi_\varepsilon(t) \varphi_\varepsilon(s) H_\varepsilon(t,s) \]

where

\[ H_\varepsilon(t,s) = \frac{1}{\varepsilon^{d+1}} \int_{-\varepsilon}^{\varepsilon} \log |x-s + t-s| \, dx \]
$$= \frac{1}{\pi} \int \int_{-\infty}^{\infty} \frac{1}{|x-y + t-s|} \, dx \, dy$$

$$= \log |t-s| + \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log |1 + (x-t)(y-s)| \, dx \, dy$$

when $x = \frac{\zeta}{|t-s|} + t$. The above integral is clearly bounded by

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{|x-y| \leq 2\delta} \log |1 + (x-t)(y-s)| \, dx \, dy$$

$$= \frac{1}{\pi} \int_{-2\delta}^{2\delta} \int_{|x+y| \leq 2\delta} \log |1 + u| \, du \, dx$$

which is bounded in turn, using

simple estimates (exercise!), by $c \log (1+x)$ for some $c > 0$.

Thus from (31.2) (31.3)

$$= \int \int_{|x+y| \leq 2\delta} \log |1 + u| \, du \, dx$$

$$\leq c \int \int_{|x+y| \leq 2\delta} \log (1 + \frac{x}{|t-s|}) \, du \, dx$$

$$\leq c \log (1 + \frac{x}{|t-s|})$$
\[ \mathcal{C} = c \left[ \int \log (\mathcal{L} - s + \varepsilon') \, \mu^\varphi_{\varepsilon'}(s) \, ds \right] - \int \| \mu^\varphi_{\varepsilon'}(s) \, ds \]. \]

As \( \log (\mathcal{L} - s + 1) \) is integrable on the (compact) support of \( \mu^\varphi_{\varepsilon'}(s) \), we conclude by monotone convergence that \( \int \| \mu^\varphi_{\varepsilon' - 1 + \varepsilon} \, \mu^\varphi_{\varepsilon'}(s) \, ds \) tends as \( \varepsilon \to 0 \) (Note: we have by (12.2), for example, \( \int \| \mu^\varphi_{\varepsilon' - 1} \, \mu^\varphi_{\varepsilon'}(s) \, ds \to \int \| \mu^\varphi_{\varepsilon'}(s) \, ds \).)

In particular, we conclude for any \( \varepsilon > 0 \), \( \varepsilon' = \varepsilon^2 \varepsilon' \) that

\[ H(\varphi_{\varepsilon'}) \leq E^\nu + \varepsilon \varepsilon' \]

where

\[ \varphi_{\varepsilon'} = \varphi_{\varepsilon'} = \varphi_{\varepsilon^2} \]

We now use the above calculus to derive a lower bound on \( Z_{\varepsilon} \). Let

\[ Z_{\varepsilon, N} = \{ x \in \mathbb{R}^N : \prod_{i=1}^N \varphi_{\varepsilon'}(x_i) > \theta^3 \}. \]
Then

\[(134.1) \quad Z_N = \int_{\mathbb{R}^N} e^{-N \sum_{i=1}^{N} V(x_i)} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \, d^N x \]

\[= \int_{\mathbb{R}^N} e^{-\xi_N(x) - \sum_{i=1}^{N} V(x_i)} \]

\[= \int_{\mathbb{E}^N} e^{-\xi_N(x) - \sum_{i=1}^{N} V(x_i)} \]

\[= \int_{\mathbb{E}^N} e^{-\xi_N - \sum_{i=1}^{N} V(x_i)} + \sum_{i=1}^{N} \left[ \sum_{j=1}^{i-1} \phi(x_i) \right] \prod_{i=1}^{N} \phi(x_i) \, d^N x \]

By Jensen's inequality,

\[\int_{\mathbb{E}^N} d\mu(x) \leq e \int_{\mathbb{E}^N} d\mu(x) \]

\[\forall f \in L^1(d\mu), \; \int d\mu = 1, \; \text{and from (134.1), we obtain} \]

\[\log Z_N \geq - \int \xi_N(x) \prod_{i=1}^{N} \phi(x_i) \, d^N x \]

\[= -N(N-1) H(\phi) - N \int V(\phi) \phi^2 \, d\mu - N \int (\log \phi(\xi)) \phi^2 \, d\mu \]

\[\geq -N^2 H(\phi) - C_2 N \geq -N^2 (E^\phi + \phi^2) - C_2 N \]
for some constant $c_\varepsilon$. Thus

\[(135.1) \quad \frac{1}{N} \log Z_N \geq -(E^V + \varepsilon)\]

for $N$ suff. large, say $N \geq N_1$. Note that $N_1$ depends only on $\varepsilon$, $N_1 = N_1(\varepsilon)$.

Now by (135.1), for $N > N_1$,

\[P_N(\mathbb{R}^N \setminus A_N, \eta + a)\]

\[\leq \frac{1}{Z_N} \int_{\{k_N(x) \leq N^+ (E^V + m + a)\}} e^{-k_N - \frac{1}{N} \sum^N_i V(x_i)} d^N x\]

\[\leq \int_{\mathbb{R}^N} e^{-\sum^N_i V(x_i)} e^{\frac{1}{2} \left( N^+ \left[ - (E^V + m + a) + E^V + \varepsilon \right] \right)} d^N x\]

\[\leq (e^{-V(\cdot) \Lambda})^N e^{N^+ (\varepsilon - m - a)}\]

\[\leq (e^{-V(\cdot) \Lambda})^N e^{-aN} e^{-\frac{m}{2} N}\]

provided we choose $\varepsilon < m/2$. Thus for $N$ suff. large, $N \geq N_1(\varepsilon)$, we have $P_N(\mathbb{R}^N \setminus A_N, \eta + a) \leq e^{-aN}$; we prove lemma 130.2.

The next lemma is the main technical result that is used to prove the convergence $\frac{1}{N} P_N(\mathbb{R}^N \setminus A_N, \eta + a)$ quick.
Lemma 136.1

Let \( \phi: \mathbb{R}^k \rightarrow \mathbb{R} \) be bounded and continuous.

Then

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{X_N} \left( e^{-\frac{1}{N^{k+1}} \sum_{i=1}^{k+1} \phi(x_{i1}, \ldots, x_{ik})} \right) = \mathbb{E} \phi(t_1, \ldots, t_k) \mathbb{E}_{\omega} \phi(t_1) \cdots \mathbb{E}_{\omega} \phi(t_k).
\]

where \( \mathbb{E}_{X_N} \) denotes expectation w.r.t. \( P_N(x_1 dX) \).

Proof: Let \( \eta > 0 \) be given and let \( X = X_{A_{N,2^k \eta}} \) be the characteristic function of the set \( A_{N,2^k \eta} \) (see previous lemma). Now

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{X_N} \left( e^{-\frac{1}{N^{k+1}} \sum_{i=1}^{k+1} \phi(x_{i1}, \ldots, x_{ik})} \right)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{X_N} \left( e^{-\frac{1}{N^{k+1}} \sum_{i=1}^{k+1} \phi(x_{i1}, \ldots, x_{ik})} X \right)
+ e^{-\frac{1}{N^{k+1}} \sum_{i=1}^{k+1} \phi(x_{i1}, \ldots, x_{ik})} (1 - X)
\]

\[
= \lim_{N \to \infty} \left( \frac{1}{N} \log \mathbb{E}_{X_N} \left( e^{-\frac{1}{N^{k+1}} \sum_{i=1}^{k+1} \phi(x_{i1}, \ldots, x_{ik})} X \right) \right)
+ \frac{1}{N} \log \frac{\mathbb{E}_{X_N} (X \mathbb{E} \phi)}{\mathbb{E}_{X_N} (X \mathbb{E} \phi)}
\]
where

\[(137.1) \quad \Phi = N^{-k+1} \sum_{x^k} \phi(x^1, \ldots, x^k).\]

As \( \phi \) is bounded, say \(|\phi| \leq b\),

\[|\Phi| \leq N^{-k+1} b N^k = bN^k.\]

Hence

\[
\exp_N \left( \frac{(1-x)e^\phi}{\exp_N (x e^\phi)} \right) = \frac{\exp_N (1-x e^bN)}{\exp_N (x e^{-bN})}
\]

\[= c^{2bN} \frac{\exp_N (1-x)}{1 - \exp (1-x)}
\]

\[\leq c^{2bN} \frac{e^{-xN^c}}{1 - e^{-xN^c}}
\]

for \(N \gg N^c\) by 130.2. Hence

\[(137.1) \quad \frac{1}{N-N^c} \log \left( 1 + \frac{\exp_N (1-x e^\phi)}{\exp_N (x e^\phi)} \right) = 0,
\]

and

\[(137.2) \quad \frac{1}{N} \log \left( \exp_N e^\phi \right) = \frac{1}{N} \log \left( \exp_N e^{\phi(x)} \right)
\]

Now the set \(A_{N,2N}\) is compact. Indeed