The remaining particles. Indeed, suppose \( F = F(x_1, \ldots, x_n) \) is a symmetric function of \( x_1, \ldots, x_n \). Then

\[
\frac{1}{n!} \int F(x_1, \ldots, x_n) \, R_n(x_1, \ldots, x_n) \, dx_1 \ldots dx_n
\]

\[
= \frac{N!}{(N-n)! \, n!} \int F(x_1, \ldots, x_n) \, P_N(x_1, \ldots, x_n, x_{n+1}, \ldots, x_N) \, dx_1 \ldots dx_{n+1} \ldots dx_N
\]

\[
= \int \sum_{1 \leq i_1 < \cdots < i_n \leq N} F(x_{i_1}, \ldots, x_{i_n}) \, P_N(x_1, \ldots, x_N) \, dx_1 \ldots dx_N.
\]

Thus

\[
(93.1) \quad \text{Exp} \: \hat{F} = \frac{1}{n!} \int F(x_1, \ldots, x_n) \, R_n(x_1, \ldots, x_n) \, dx_1 \ldots dx_n
\]

where

\[
(93.2) \quad \hat{F}(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_n \leq N} F(x_{i_1}, \ldots, x_{i_n})
\]

is the symmetric extension of \( F(x_1, \ldots, x_n) \) to \( N \) variables.

Suppose \( x_1^0 < x_2^0 < \cdots < x_n^0 \) and let \( \delta > 0 \) be small.

Let \( \chi_D \) be the characteristic function of the disjoint
Let 

\[ F(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} x_j^0 \left( \frac{\delta_j + 1}{2} \right) \]

where \( F \) is symmetric.

From (93.2) we have

\[ \delta^n R_n(x_1^0, \ldots, x_n^0) \sim \frac{1}{n^N} \int F(x_1, \ldots, x_n) P_n(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \]

\[ = \int \hat{F}(x_1, \ldots, x_n) \hat{P}_n(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \]

(whence \( \hat{P}_n = N! P_n \))

\[ = \sum_{x_1 \prec \cdots \prec x_n} \hat{F}(x_1, \ldots, x_n) \hat{P}_n(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \]

Now \( x_1 \prec \ldots \prec x_n \), \( F(x_1, \ldots, x_n) = \prod_{j=1}^{n} x_j^0 \left( \frac{\delta_j + 1}{2} \right) \). Hence

\[ \delta^n R_n(x_1^0, \ldots, x_n^0) = \sum_{x_1 \prec \cdots \prec x_n} \left[ \sum_{1 \leq i_1 < \cdots < i_N} \left( \prod_{j=1}^{n} x_j^0 \left( \frac{\delta_j + 1}{2} \right) \right) \right] \hat{P}_n(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \]

\[ = \sum_{x_1 \prec \cdots \prec x_n} \left[ (x_1^0(x_1) \cdots x_n^0(x_n) + \cdots + x_n^0(x_{n+1}) \cdots x_N^0(x_N)) \hat{P}_n(x_1, \ldots, x_n) \right] \, dx_1 \cdots dx_n \]

\[ \sim \text{Prob} \text{ (exactly 1 eigenvalue in each of } [0,1] \text{ intervals } (x_1^0 - \delta/2, x_1^0 + \delta/2) \text{)} \]

why?
Note: Insert on page.

In the case of RMT, $P_N(\lambda_1, \ldots, \lambda_N) d\lambda_1 \cdots d\lambda_N$
only has physical meaning, even though $P_N(\lambda_1, \ldots, \lambda_N)$ is
symmetric in the $\lambda_i$, only when $\lambda_1 \leq \cdots \leq \lambda_N$. Indeed,
remember that the map $m \mapsto (\Lambda(m), \sigma(m))$ always
specifies the eigenvalues in some order, in particular, $\lambda_1(m) \leq \cdots \leq \lambda_N(m)$.

When we compute the expectation $\mathbb{E}\exp \mathcal{F}$ for
some quantity $\mathcal{F}(\lambda_1, \ldots, \lambda_N)$ which is symmetric in $\lambda_1, \ldots, \lambda_N$,
we have

\begin{equation}
\mathbb{E}\exp \mathcal{F} = \int \mathcal{F}(\lambda_1, \ldots, \lambda_N) P_N(\lambda_1, \ldots, \lambda_N) d\lambda_1 \cdots d\lambda_N
\end{equation}

However, as a computational convenience, we observe that

\begin{equation}
\mathbb{E}\exp \mathcal{F} = \frac{1}{N!} \int \mathcal{F}(\lambda_1, \ldots, \lambda_N) P_N(\lambda_1, \ldots, \lambda_N) d\lambda_1 \cdots d\lambda_N
\end{equation}

Although (94.1) is easier to manipulate, when we want to
understand the meaning of the statistic $\mathbb{E}\exp \mathcal{F}$, we must refer to
(94.1).
Thus

\[(95.1)\]
\[R_n(x_1, \ldots, x_n) \text{ is the density of the probability}\]

that is one eigenvalue at each of the points

\[x_1, \ldots, x_n, \ x_1 < \ldots < x_n\]

Note the following:

If \( F = F(x_i) = \chi_x(x_i) \), the characteristic function.

\( \forall x \in \mathbb{R} \)

\[\hat{F}(x_1, \ldots, x_N) = \sum_{i=1}^{N} F(x_i) = \sum_{i=1}^{N} \chi_x(x_i) = \# \{ i : x_i \in A \} \]

Thus by (93.1)

\[\exp \left( \# \{ i : x_i \in A \} \right) = \int_{A} R(x) \, dx\]

Bearing (94.1) in mind, we also define for random matrix ensembles,

\[\exp \left( \# \{ x_i \in A \} \right) = \int_{A} R(x) \, dx\]

Also if \( A_1, A_2 \) are two disjoint sets in \( \mathbb{R} \) can

\[F(x_1, x_2) = \chi_{A_1}(x_1) \chi_{A_2}(x_2) + \chi_{A_1}(x_2) \chi_{A_2}(x_1)\]

Then

\[\hat{F}(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < i_2 \leq N} \left[ \chi_{A_1}(x_{i_1}) \chi_{A_2}(x_{i_2}) + \chi_{A_1}(x_{i_2}) \chi_{A_2}(x_{i_1}) \right] \]
\[ = \# \{ (i_1, i_2) : i_1 < i_2, (x_{i_1}, x_{i_2}) \in \Lambda_1 \times \Lambda_2 \cup \Lambda_2 \times \Lambda_1 \} \]

Thus

\[(96.1) \quad \text{Exp} \left( \sum_{\text{pair}} (i_1, i_2) \right) = \frac{1}{2!} \int \left[ \chi_{\Lambda_1}(x_1) \chi_{\Lambda_2}(x_2) + \chi_{\Lambda_2}(x_1) \chi_{\Lambda_1}(x_2) \right] \text{det} \left( \Lambda(x_1, x_2) \right) \, dx_1 \, dx_2.

\]

\[= \int_{\Lambda_1 \times \Lambda_2} R(x_1, x_2) \, dx_1 \, dx_2.\]

Exercise: Show how (96.1) changes if \( \Lambda_1 = \Lambda_2 \).

We now show how to compute \( R_n(x_1, \ldots, x_n) \) using

\[(99.1) \quad \langle \mathbb{F} \rangle = \text{det} \left( I + (L^2 + K x g) \right) \]

where \( f(m) = \text{det} (I + g(m)) \) and \( K \) is given in (88.2).
Again bearing (4.1) in mind, we have for random matrix ensemble,

\[ \int_{x_1 < \ldots < x_N} \hat{F}(x_1, \ldots, x_N) \, \hat{p}_N(x) \, d^{N}x. \]

\[ = \int_{1 \leq i_1 < i_2 \leq N} \hat{F}(x_{i_1}, \ldots, x_{i_2}) \, p_N(x) \, d^{N}x, \quad \hat{p}_N = \frac{1}{N!} \hat{p}_N. \]

\[ = \int_{\lambda_1 \times \lambda_2} R(x_1, x_2) \, dx_1 \, dx_2 \]

(96.1)

For definiteness that

Now suppose \( \lambda_1 \) lies to the left of \( \lambda_2 \)

\[ \lambda_1 \quad \lambda_2 \]

Then for \( x_1 < \ldots < x_N \)

\[ \hat{F}(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < i_2 \leq N} X_{i_1}(x_{i_1}) \, X_{i_2}(x_{i_2}) \]

\[ = \# \ \text{of ordered pairs of eigenvalues}, \]

\[ (x_{i_1}, x_{i_2}), \quad x_{i_1} < x_{i_2} \]

such that \( (x_{i_1}, x_{i_2}) \in \lambda_1 \times \lambda_2 \)

\[ \text{Here} \]

\[ \exp \oint \oint \# \ \text{of ordered pairs of eigenvalues}, \]

\[ (x_{i_1}, x_{i_2}), \quad x_{i_1} < x_{i_2}, \]

\[ = \int_{\lambda_1 \times \lambda_2} R(x_1, x_2) \, dx_1 \, dx_2, \]
More explicitly for any \( g \in L^\infty(\mathbb{R}^N) \),

\[
(97.0) \quad \int \frac{1}{\prod_{i=1}^N (1 + g(x_i))} P_N(x_1, \ldots, x_N) \, d^N x = \det (1 + k X_h)_{L^1(\mathbb{R}^N)} = 1
\]

where

\[
P_N(x) \, dx = \frac{\prod_{i=1}^N w(x_i)}{\sqrt{\prod_{i=1}^N \Big| V(x_i) \Big|^2}} \, d^N x
\]

Choose \( g \) such that

\[
1 + g = \sum_{i=0}^k \lambda_i x_i \quad \text{for some } k
\]

where \( x_i \) are the characteristic functions of disjoint sets \( \Lambda_i \), \( 0 \leq i \leq k \), in \( \mathbb{R}^N \), such that \( \bigcup_{i=0}^k \Lambda_i = \mathbb{R}^N \).

Here \( \lambda_i \in \mathbb{R} \), \( i=0, \ldots, k \).

Clearly

\[
g = (\delta_{01} - 1) \lambda_0 x_0 + \cdots + (\delta_{k1} - 1) \lambda_k x_k
\]

\[
= \sum_{i=0}^k m_i x_i, \quad m_i = \delta_{i1} - 1, \quad 0 \leq i \leq k.
\]

For any \( 1 \leq i \leq N \), let

\[
(97.2) \quad \delta_i (z_1, \ldots, z_N) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_N} z_{i_1} \cdots z_{i_N}
\]

denote the \( i \)-th elementary symmetric function and set \( \delta_0 = 1 \).
We have

\[ (98.1) \quad \prod_{i=1}^{N} (1 + g(x_i)) = \sum_{j=0}^{\infty} \sigma_j (\xi_1, \ldots, \xi_N) \]

Thus

\[
\int \prod_{i=1}^{N} (1 + g(x_i)) \, P_N(x) \, d^N x \\
= \sum_{j=0}^{\infty} \sum_{1 \leq i_1 < \ldots < i_j \leq N} \int g(x_{i_1}) \cdots g(x_{i_j}) \, P_N(x) \, d^N x \\
= \sum_{j=0}^{\infty} \binom{N}{j} \int g(x_1) \cdots g(x_j) \, P_N(x) \, d^N x
\]

(by symmetry)

\[ = \sum_{j=0}^{\infty} \binom{N}{j} \frac{(N-j)!}{N!} \int g(x_1) \cdots g(x_j) \, R_j(x_1, \ldots, x_j) \, dx_1 \cdots dx_j \]

(Insert 98.1, 98.2)

Substitute \((88.1, 88.2)\) for \(\prod_{i=1}^{N} (1 + g(x_i))\) we find (exercise: see ref(3), p87)

\[ (98.2) \quad \int \prod_{i=1}^{N} (1 + g(x_i)) \, P_N(x) \, d^N x \\
= \sum_{n_0, n_1, \ldots, n_k \geq 0 \atop 0 \leq \ln_1 \leq N} \frac{m_0 \cdot m_1 \cdots m_k}{\ln_1!} \int R_{\ln_1}(x_1, \ldots, x_{\ln_1}) \times
\]

\( R_{\ln_1} \) for \( x_{\ln_1, \ldots, \ln_k} \) \{ x_1, \ldots, x_{\ln_1} \} \) lie in \( N_0, N_1, \ldots, N_k \) resp \( d^N x \)
Now

\[(9.8+1) \prod_{i=1}^{j} g_i(x_i) = \prod_{i=1}^{j} \left( \eta_i X_{i1}(x_i) + \cdots + \eta_i X_{ik}(x_i) \right) \]

\[= \sum_{\substack{\mathbf{n} \geq 0 \\mathbf{n_i} \geq 0 \\mathbf{n_k} \geq 0}} \sum_{i=0}^{\mathbf{n_k}} \epsilon_i = 0 \\epsilon_j = \mathbf{n_0} \\epsilon_k = \mathbf{n_k} \]

\[E(\mathbf{n_0}, \ldots, \mathbf{n_k}; x) \]

where

\[(9.8+2) \quad E(\mathbf{n_0}, \ldots, \mathbf{n_k}; x) = \sum_{0 \leq i, \ldots, i_j \leq k} X_{i1}(x_i) \cdots X_{ik}(x_i) \]

Consider, for example, the case where \(k = 5\) and \(j = 6\) and

\[\mathbf{n_0} = 1 \quad \mathbf{n_1} = 0 \quad \mathbf{n_2} = 2 \quad \mathbf{n_3} = 1 \quad \mathbf{n_4} = 0 \quad \mathbf{n_5} = 2 \quad \sum_{d=0}^{6} \mathbf{n_d} = \mathbf{j} = 6 \]

and with \((x_1, \ldots, x_6)\) arranged as follows

\[x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \]

Now clearly, \(X_{i1}(x_i) X_{i2}(x_i) \cdots X_{i6}(x_i) = 1 \) if and only

\[i_1 = 2 \quad i_2 = 3 \quad i_3 = 5 \quad i_4 = 2 \quad i_5 = 0 \quad i_6 = 5 \]
In particular, only one term in (98+2.7) contributes. We conclude that

\[(98+1) \quad E(n_0, n_1, \ldots, n_k; x) = \sum_{\{x_1, \ldots, x_j\} : \eta_q \notin \{x_1, \ldots, x_j\}} \sum_{\eta_q \notin \{x_1, \ldots, x_j\}} \eta_1 \cdot \cdots \cdot \eta_k} \tag{x} x = (x_1, \ldots, x_j) : \eta_q \notin \{x_1, \ldots, x_j\}, 0 \leq q \leq k \]

Thus,

\[(8+1.2) \quad \prod_{i=1}^{j} f(x_i) = \sum_{\eta_0, \eta_1, \ldots, \eta_k \geq 0} \eta_0^{\eta_1} \cdot \cdots \cdot \eta_k^{\eta_k} \sum_{\{x_1, \ldots, x_j\} : \eta_q \notin \{x_1, \ldots, x_j\}} \sum_{\eta_q \notin \{x_1, \ldots, x_j\}} \eta_1 \cdot \cdots \cdot \eta_k} \tag{y} x = (x_1, \ldots, x_j) : \eta_q \notin \{x_1, \ldots, x_j\}, 0 \leq q \leq k \]
On the other hand, by the Fredholm expansion of a determinant,

\[
det (1 + k_g) = \sum_{j=0}^{\infty} \frac{1}{j!} \int_{\mathbb{R}^j} \det \begin{bmatrix} k(x_i, x_1) & \cdots & k(x_i, x_j) \\ \vdots & \ddots & \vdots \\ k(x_i, x_{i-1}) & \cdots & k(x_i, x_j) \end{bmatrix} \prod_{i=1}^{j} g(x_i) \, dx_i \\
= \sum_{j=0}^{N} \frac{1}{j!} \int_{\mathbb{R}^j} \det (k(x_i, x_j)) \prod_{i=1}^{j} g(x_i) \, dx_i \\
\text{why?}
\]

Again expanding out \( g(x) \) using (98.2), we find as above

\[
(99.1) \quad \det (1 + k_g) = \sum_{\mathbf{n}} \frac{m_0 \cdots m_N}{n_0! \cdots n_N!} \int_{\mathbb{R}^n_0} \cdots \int_{\mathbb{R}^n_N} \det (k(x_i, x_j)) \prod_{i=1}^{N} g(x_i) \, dx_i 
\]

\[
\times \chi_{\{n_0, n_1, \ldots, n_N \text{ of } \{x_1, \ldots, x_N\} \text{ lie in } n_0, n_1, \ldots, n_N \}} \, dx_i
\]

Equating (98.2) and (99.1), and comparing coefficients, we find in particular for \( k \leq N, n_0 = 0, n_1 = \ldots = n_k = 1 \).
\[ (100.1) \quad 0 = \int_{\mathbb{R}^k} \Delta_h(x_1, \ldots, x_k) \, d^k x \]

where \( \Delta_h(x_1, \ldots, x_k) = R_h(x_1, \ldots, x_k) - \det(K(x_i, x_j)) \) for one of \( x_i, \ldots, x_k \), no other none.

As \( \Delta_h(x_1, \ldots, x_k) \) is symmetric, (100.1) =

\[ (100.2) \quad \int_{x_1 \leq \cdots \leq x_k} \Delta_h(x_1, \ldots, x_k) \, d^k x = 0 \]

\( \# \{ i : x_i \notin (a, b) \} = 1 \)

\( 1 \leq i \leq k \)

Let \( \lambda_i = (a_i, b_i) \), \( 1 \leq i \leq k \), be disjoint intervals ordered from the left \( u \)

\[ a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k \]

and inserting these \( \lambda_i \)'s into (100.2) and letting \( b_i \downarrow a_i \) as we obtain

\[ \Delta_h(a_1, \ldots, a_k) = 0 \]

for all \( a_1, \ldots, a_k \) and hence for all \( a_1, \ldots, a_k \) by symmetry. We conclude that for \( k > 1 \),

\[ (100.3) \quad R_h(x_1, \ldots, x_k) = \det(K(x_i, x_j)) \prod_{1 \leq i, j \leq k} \]

\[ i \neq j \]
Exercise: Use the above calculation to derive (95.1)

Remark: For other proofs of this result see ref 3 pp 96-98

and also ref 2 pp 103-104 (This calculation is taken from [Meh].)

The above calculations show that in order to evaluate key statistics for unitary ensembles we must understand the asymptotic behavior of the correlation kernel

$$K(x, y) = \sum_{j=0}^{N-1} \phi_j(x) \phi_j(y)$$

where $\phi_j(x) = p_j(x) (w(x))^2$, and the $p_j$'s are orthonormal with the weight $w(x)$,

$$\int_{\mathbb{R}} p_j(x) p_k(x) w(x) \, dx = \delta_{ij}, \quad 0 \leq i, j \leq \infty.$$

Thus the problem of the asymptotics of eigenstatistics reduces, for unitary ensembles, to the classical problem of the asymptotics of orthogonal polynomials (OP's).
We now compute

$$\text{Prob} \left( n_1 \text{ eigenv. in } \lambda_1, \ldots, n_k \text{ eigenv. in } \lambda_k \right)$$

where again the $\lambda_i$ are disjoint and $\sum_{j=1}^{k} n_j = N.$

Set $n_0 = N - \sum_{j=1}^{k} n_j$ and set $\Lambda_0 = \mathbb{R} \setminus \bigcup_{j=1}^{k} \Lambda_j.$

Again letting $\chi_i$ be the characteristic function of $\Lambda_j,$ $e_{\chi_i} \in k,$ we have, using (48.2.1)

$$\text{Prob} \left( n_1 \text{ eig.'s in } \Lambda_1, \ldots, n_k \text{ eig.'s in } \Lambda_k \right)$$

$$= \int_{\chi_1 \leq \ldots \leq \chi_N} E(n_0, n_1, \ldots, n_k; x) \prod_{i=0}^{N} w(x_i) \left( V(x) \right)^2 d^N x$$

$$= \int_{\chi_1 \leq \ldots \leq \chi_N} \frac{\prod_{i=0}^{N} w(x_i)}{\left( V(x) \right)^2} d^N x$$

$$= \int_{\mathbb{R}^N} E(n_0, n_1, \ldots, n_k; x) \prod_{i=0}^{N} w(x_i) \left( V(x) \right)^2 d^N x$$

Now for

$$F(x, x_0, \ldots, x_k) = \prod_{j=1}^{N} (\sum_{i=0}^{k} \chi_i X_j) \left( x_1 \right)$$

we have for $\Sigma_i n_i < N$

$$E(n_0, n_1, \ldots, n_k; x) = \frac{\partial^{n_1 \ldots n_k}}{\partial x_1^{n_1} \ldots \partial x_k^{n_k}} \left|_{x_1 = \ldots = x_k = 0} \right| E(x : x_0, \ldots, x_k)$$
where we have used \( q_i = n_i \) with \( \sum_i n_i = N \)

\[
F = \frac{1}{N!} \left( \delta_0 x_0 + \cdots + \delta_N x_N \right)(x)
\]

\[
= \sum_{n_0, n_1, \ldots, n_N \geq 0} \frac{\delta^{n_0} \cdots \delta^{n_N}}{\partial \delta_0^{n_0} \cdots \partial \delta_N^{n_N}} \left| \begin{array}{c}
\delta_0 = 1 \\
\delta_1 = \cdots = \delta_N = 0
\end{array} \right|
\]

Thus

\[
\prod \left( n_i \right) = \frac{\partial^{n_1 + \cdots + n_N}}{\partial \delta_1^{n_1} \cdots \partial \delta_N^{n_N}} \left| \begin{array}{c}
\delta_0 = 1 \\
\delta_1 = \cdots = \delta_N = 0
\end{array} \right|
\]

\[
J[n] \propto \int \frac{1}{N!} \left( \delta_0 x_0 + \cdots + \delta_N x_N \right)(x) \prod_{i=1}^{N} \frac{\partial^{n_i}}{\partial \delta_i^{n_i}} \left| \begin{array}{c}
\delta_0 = 1 \\
\delta_1 = \cdots = \delta_N = 0
\end{array} \right|
\]

\[
\int e^{-\int \frac{1}{N!} \left( \delta_0 x_0 + \cdots + \delta_N x_N \right)(x) \prod_{i=1}^{N} \frac{\partial^{n_i}}{\partial \delta_i^{n_i}} \left| \begin{array}{c}
\delta_0 = 1 \\
\delta_1 = \cdots = \delta_N = 0
\end{array} \right|}
\]

Set \( \mathbf{g} = \delta_0 x_0 + \cdots + \delta_N x_N \) and hence at \( \delta_0 = 1 \),

\[
g = n_1 x_1 + \cdots + n_N x_N \quad \text{where} \quad n_i = \delta_i - 1, \; 1 \leq i \leq N
\]

It follows now from (89.11)

\[
< f > = < 1 + g > = \det \left( I + N x g \right)
\]

where \( \mathbf{K} \) is the correlation kernel in (88.21)

\[
K(x, y) = \sum_{i=0}^{N} \phi_i(x) \phi_i(y)
\]
Thus, finally, \( (101+++) \) \( \prod \) s.e.s. \( \alpha_1, \ldots, \alpha_k \) express in \( \lambda_1, \ldots, \lambda_n \)

\[
= \left| \begin{array}{c}
\eta_1' & \ldots & \eta_n' \\
\eta_1'' & \ldots & \eta_n'' \\
\eta_1''' & \ldots & \eta_n''' \\
\end{array} \right|, \quad \det \left( 1 + k \sum_{i=1}^{K} \eta_i x_i \right)
\]
(Ref: Szegő, "Orthogonal Polynomials").

For the next couple of lectures we will consider this problem. A key object that controls the asymptotics of OP's is the so-called equilibrium measure (see Ref 2, Chap 6; see also Saff & Totik "logarithmic potentials and external fields" for the general theory).

We will see eventually that this quantity is intimately related to the density of states for RMT and also to the one-point correlation function \( R_1(x) \). The calculations below are taken from Ref (2), which in turn are based on work of K. Johansson (see Ref (2)). In the calculations that follow we will always assume that the probability density \( p_n(x) \) defined varies with \( N \) in the following way:

\[
(102.1) \quad p_n(x)dx = \frac{1}{Z_N} e^{ -N + V(x) } dx
\]

As all our calculations so far have assumed that \( N \) is given and fixed, they all remain valid: we
must just set
\[ w = w_n = e^{-N + V(x)} \]

After integrating out the eigenvectors we obtain as before a measure on the eigenvalues
\[ \hat{P}_N(\lambda) d\lambda = \frac{1}{Z_N} e^{-N \sum V(\lambda_i)} \prod_{i \neq j} (\lambda_i - \lambda_j)^2 d\lambda \]

where (after symmetrizing)

\[ Z_N = \int_{\mathbb{R}^N} e^{-N \sum V(\lambda_i)} \prod_{i \neq j} (\lambda_i - \lambda_j)^2 d\lambda \]

\[ = \int_{\mathbb{R}^N} e^{-N^2 H(\lambda)} d\lambda \]

where \[ H(\lambda) = \frac{1}{N^2} \sum_{i \neq j} \log |\lambda_i - \lambda_j|^{-1} + \lambda \sum_{i} V(\lambda_i) \]

Let
\[ M_\lambda = \frac{1}{N} \sum_{i} \delta_{\lambda_i} \]

be the normalized counting measure for the eigenvalues.

and note that \[ H(\lambda) \] can be expressed as
follows:

\[(104.1) \quad H(\mu_\lambda) = \int \log \left| \tilde{\xi}^{-1} q_{\lambda}(t) q_{\lambda}(s) \right| + \int \nu(t) q_{\lambda}(t) \]

Note that the scaling in the potential

\[e^{-tN} \rightarrow e^{-NtN} \]

is chosen so that the terms in (104.1) are balanced.

Intuitively, the leading contributions to the partition function as \(N \rightarrow \infty\)

\[(103.1) \quad \chi_{\lambda} \] comes from those \(\xi^s\) for which \(H(\mu_\lambda)\)

is a minimum. Thus we are led to consider the following minimization problem

\[(104.2) \quad E^V = \inf_{\mu \in M_1(\mathbb{R})} H(\mu) \]

where \(H(\mu) = \int \log \left| \tilde{\xi}^{-1} q_{\lambda}(t) q_{\lambda}(s) \right| + \int \nu(t) q_{\lambda}(t) \)

and \(M_1(\mathbb{R}) = \mathcal{M}_{\mu} \) is a Borel measure on \(\mathbb{R}: \int q_{\lambda} = 1\).
We will show eventually that (103.2) has a unique minimizer \( \mu = \mu^{eq} \), the equilibrium measure mentioned above. From the definition of \( H(\mu) \), \( \mu^{eq} \) has an electrostatic interpretation: it is the equilibrium configuration for electrons with logarithmic electrostatic repulsion

\[
\int \log(1 - s \cdot \phi^{eq}(t) \phi^{eq}(s))
\]

in an external field \( \int v(t) \phi^{eq}(t) \). As already indicated, \( \phi^{eq} \) is intimately related to a variety of problems in RMT, and also in analysis. The existence and uniqueness of the solution of the variational problem (104.2) relies ultimately on the fact that we are dealing with a (constrained) convex minimization problem.