Proof: Recall that $\mu^*$ has compact support and $H(\mu^*) = E^V < \infty$. Suppose $\tilde{\mu} \in M_1(\mathbb{R})$ has compact support and $H(\tilde{\mu}) < \infty$. Then writing

$$\mu_t = t \tilde{\mu} + (1-t) \mu^* = \mu^* + t(\tilde{\mu} - \mu^*) \in M_1(\mathbb{R}), 0 \leq t \leq 1,$$

we have

$$(15.2.1) \quad H(\mu_t) = \int \log |x-y| \, q^{eq}(x) \, q^{eq}(y) + \int V(x) \, q^{eq}(x)$$

$$+ 2t \int \log |x-y| \, q^{eq}(x) \, d(\mu - \mu^*) \, (x)$$

$$+ t \int V(x) \, d(\mu - \mu^*) \, (x)$$

$$+ t^2 \int \log |x-y| \, q^{eq}(x) \, d(\mu - \mu^*) \, (x) \, d(\tilde{\mu} - \mu^*) \, (y).$$

All these steps are justified as $\mu^*$, $\tilde{\mu}$ have compact support and $H(\mu^*), H(\tilde{\mu}) < \infty$.

As $H(\mu_t) > H(\mu^*), 0 < t < 1$, it is clear that necessary that

$$\int \left[ 2 \int \log |x-y| \, q^{eq}(x) + V(x) \right] d(\tilde{\mu} - \mu^*) \, (x) = 0.$$
(153.1) \[ \int [2 \int_{y_1}^{y_2} q_u(x, y) + U(x)] q_u(x) \; dx \geq \epsilon \]

where

\[ \epsilon = \int [2 \int_{y_1}^{y_2} q_u(y) + U(x)] u^d(x) \]

This proves (i).

Let

\[ B = \{ x : \epsilon \int_{y_1}^{y_2} q_u(y) + U(x) < \epsilon \} \]

Clearly, \( B \) is a bounded set as \( \frac{V(x)}{\log(x+1)} \)

as \( x \to 0^+ \). Now suppose \( \tilde{\mu}(B) > 0 \) and set

(153.3) \[ \tilde{\mu}_B = \frac{x_B \tilde{\mu}}{\tilde{\mu}(B)} \]

where \( x_B \) is the characteristic function of the set \( B \). Then \( \tilde{\mu}_B \in M_1(\mathbb{R}) \) and when compact support, \( \tilde{\mu}_B \in M_1(\mathbb{R}) \) and when compact support, and \( H(\tilde{\mu}_B) < \infty \) (why?). Inserting \( \tilde{\mu}_B \)

into (153.1) we find
\[
\lambda \leq \int \left[ 2 \int \log(x-\eta^{-1}) \phi_{\mu^0}(u) + V(x) \right] \frac{\nu_B(x)}{\delta^a(x)} \, dx < \infty
\]

which is a contradiction. Hence

\[(154.1) \quad \tilde{\mu}(B) = \tilde{\mu} \left( \{ x : 2 \int \log(x-\eta^{-1}) \phi_{\mu^0}(u) + V(x) < \epsilon \} \right) = 0
\]

for all measures \( \tilde{\mu} \) with compact support and \( H(\tilde{\mu}) < \infty \), and in particular for \( \tilde{\mu} = \mu^{eq} \).

But from (153.2)

\[
0 = \int \left[ 2 \int \log(x-\eta^{-1}) \phi_{\mu^0}(u) + V(x_1-x) \right] \phi_{\mu^{eq}}(x)
\]

Any (154.1), and so \( \alpha \) follows.

Conversely, suppose \( \mu \in M_{1,1}(\Omega) \), satisfies \( \alpha(i + \alpha) \) and

\[ H(\mu) < \infty \]

and \( \mu \) has compact support. Then write

\[ \mu^{eq} = \mu + (\mu_{\mu} - \mu) \]

As in (152.1), we find

\[ H(\mu) \geq H(\mu^{eq}) \]

\[ = H(\mu) + \int \left[ 2 \int \log(x-\eta^{-1}) \phi_{\mu^0}(u) + V(x_1) \right] \phi_{\mu^{eq}-\mu}(x)
\]

\[ + \int \log(x-\eta^{-1}) \phi_{\mu^{eq}-\mu}(x) \phi_{\mu^{eq}-\mu}(x) \]
By (i) and (ii) the second term reduces to
\[
\sqrt{2 \log (1 - x_1 \Phi(1) + u(1)))} \rho \equiv -\ell
\]
\[\geq \ell - \ell = 0\]

But the third term is strictly positive (see (126.11))

unless \( m^\equiv = \mu \). On the other hand, if \( m^\equiv \neq \mu \)

then the above calculation shows that

\[ H(\mu) \geq H(\mu^\equiv) > H(\ell), \]

a contradiction. Thus if \( \mu \) satisfies (i) and (ii) above, it

must be \( m^\equiv \). \( \square \).

If we know a priori that \( \rho^\equiv = \rho(x) dx \) for some

(positive) continuous function \( \rho \), say, of compact support,

then \( z / \log (1 - x_1 \Phi(1) + u(1)) \) is continuous and

\( (m, 1) \) takes the following strange form:
Thm 156.1 (Variational setting: strong form).

Suppose \( q = q(x) \) for some cont. (pos.) func. 4 of compact supp. Then (i) (ii) above can be replaced by

\[ (i)' \quad 2 \int_{\mathbb{R}} |x-y|^{-1} q^{|x|}(y) + U(x) = \mu(x) \quad \forall x \]

\[ (ii)' \quad 2 \int_{\mathbb{R}} |x-y|^{-1} q^{|y|}(y) + U(x) = \mu(x) \quad \text{on} \quad 4x: 4(x) > 0 \]

We now show how to use Thm 156.1 to compute \( \mu \) for the case \( U(x) = \mu x^2 \), \( m \geq 0 \).

We need \( \mu \) in the form

\[ (156.2) \quad \mu = 4(x) \]

with \( \int_{4(x)} = 1 \)

for some continuous \( 4(x) > 0 \) of compact support, which satisfies (i)', (ii)' above. If we succeed in producing such a \( 4(x) \), then \( \mu = 4(x) \) is necessarily \( \mu \) by Thm 156.1.
Now observe that the weak derivative of $f(x)$ is given by

$$F(x) = 2\int |y| |\psi(x-y反号)\| \chi(y) dy$$

$D F(x) = -2\int \phi(x) \int |y| |x-y反号| \psi(|y|) dy, \phi \in \mathcal{D}(\mathbb{R})$

$$= -\lim_{\varepsilon \to 0} \int \phi(x) \int |y| |x-y反号| \psi(|y|) dy$$

(by dominated convergence: indeed as $\psi, \phi$ have compact support and $\int$)

$$\leq \int |y| |x-y反号| dy + \int |y| dy \leq \int |y| |x-y反号| dy + \int |y| dy$$

$$\leq 0 \leq x \leq 2 \pi$$

$$= \lim_{\varepsilon \to 0} \int \phi(x) \int |y| |x-y反号| \psi(|y|) dy$$

$$= -\int \phi(x) \int |y| |x-y反号| \psi(|y|) dy, \phi \in \mathcal{D}(\mathbb{R})$$

We have

$$(157.1) \qquad \int \frac{4(y)}{x-y} dy = \frac{1}{11} \int \frac{4(y)}{x-y} dy$$
to the Hilbert transform of $h$. Here we have used the fact that $h \in C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ and

$$
\frac{1}{i\pi} \int \frac{x-y}{(x-y)^2 + \varepsilon^2} \, h(y) \, dy \to Hh(x)
$$

in $L^1(\mathbb{R})$ (see, e.g., [Y. Katznelson], *An introduction to harmonic analysis*).

It follows that $F$ has a distributional derivative in $L^2(\mathbb{R})$ and from (ii) we must have from (ii)

$$
(158.1) \quad -2\pi i H h(x) + H h'(x) = 0 \quad \text{a.e. on } \{4|x| > 0\}.
$$

Define the Borel transform $G$ of $h$ by

$$
(158.2) \quad G(z) = \frac{1}{i\pi} \int \frac{h(y) \, dy}{y - z}, \quad z \in \mathbb{C} \setminus \text{supp } h.
$$

Note that $G$ is analytic on $\mathbb{C} \setminus \text{supp } h$. Now by standard theory (see again [Katznelson]) the limits

$$
(158.3) \quad G_+^\varepsilon(x) = \lim_{\varepsilon \to 0^+} \int \frac{h(y) \, dy}{y - (x + i\varepsilon)}
$$

and

$$
(158.4) \quad G_-^\varepsilon(x) = \lim_{\varepsilon \to 0^+} \int \frac{h(y) \, dy}{y - (x - i\varepsilon)}
$$
exists in $L^2(\mathbb{R}, d\mu)$, and also pointwise a.e., and

\[(15a.1)\quad G_\pm(x) = \pm 4(x) + i \, H_4(x)\]

We learn that, a.e., on $4(x) > 0$

\[(15a.2)\quad G_+(x) + G_-(x) = 2i \, H_4(x) = i \, \nu'(x)\]

Combining this with

\[(15a.3)\quad G(1_3) \to 0 \quad \text{as} \quad 3 \to \infty, \quad \text{we}

see that \((15a.2)\) \& \((15a.3)\) give a \underline{real} Riemann-Hilbert problem \((\text{RHP})\) for $G$, and hence for $4$. This RHP is \underline{not in standard form} because of the sum $G_+ + G_-$ in \((15a.2)\) rather than the difference. In special cases, however, this can be converted into a standard RHP. Suppose, for example, that

\[(15a.4)\quad \left\{ \begin{array}{l}
\nu(x) > 0 \text{ consist of a finite #}
\text{of (disjoint) intervals} \\
(4(x) > 0) \quad \text{a.e.} \quad (a_i, b_i) = \Sigma
\end{array} \right.\]
Let
\[ q(z) = \prod_{i=1}^{k} (3 - a_i \cdot x - b_i) \]
and define \( (q(z))^{\frac{1}{2}} \) as an analytic function in \( C \setminus \Sigma \) so that
\[ (q(z))^{\frac{1}{2}} \sim +z^k \quad \text{as} \quad z \to \infty \]

Set
\[ \tilde{G}(z) = \frac{G(z)}{(q(z))^{\frac{1}{2}}} \]

Then for \( z \in \mathbb{C}\setminus\{0\} \) we have
\[ (q(z))_{+}^{\frac{1}{2}} = - (q(z))_{-}^{\frac{1}{2}} \]

Hence for \( z \in \mathbb{C}\setminus\{0\} \)

\[ \tilde{G}_{+}(z) - \tilde{G}_{-}(z) = \frac{G_{+}(z)}{(q(z))_{+}^{\frac{1}{2}}} - \frac{G_{-}(z)}{(q(z))_{-}^{\frac{1}{2}}} = \left[ G_{+}(z) + G_{-}(z) \right] / (q(z))_{+}^{\frac{1}{2}} = \frac{i}{a} \frac{U'(z)}{(q(z))_{+}^{\frac{1}{2}}} \]
On the other hand \((q(3))^{1/2}\) is analytic
in \(\mathbb{C} \setminus \bar{\Sigma}\) and hence

\[(161.1) \quad \tilde{G}_+(z) - \tilde{G}_-(z) = 0, \quad z \in \mathbb{R} \setminus \bar{\Sigma}\]

so that \(\tilde{G}(z)\) is analytic in \(\mathbb{C} \setminus \Sigma\)

and also, clearly,

\[(161.2) \quad \tilde{G}_-(z) \to 0 \quad \text{as} \quad z \to \infty\]

Hence we are lead to a standard RHP

\[(161.3)(161.2)\] on \(\Sigma\), which can be solved by

the Plancherel formula

\[(161.3) \quad \tilde{G}_-(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{\frac{i}{\pi} \sqrt{1/s}}{(q(3))^{1/2} + s-z} \, ds\]

Indeed, by \((159.1)\)

\(\tilde{G}_+(z) - \tilde{G}_-(z) = \frac{1}{2} \left[ \frac{i}{\pi} \sqrt{1/s} - (-\frac{i}{\pi} \sqrt{1/s}) \right] = \frac{i}{\pi} \sqrt{1/s} \quad \text{for} \quad s \in \Sigma.\)

And clearly \(\tilde{G}(z)\) is analytic

in \(\mathbb{C} \setminus \Sigma\) as \(\tilde{G}(z) \to 0 \quad \text{as} \quad z \to \infty.\)

We thus have
(162.1) \[ G(z) = \left( \frac{g(3)}{2\pi i} \right)^{\frac{1}{2}} \sum \int \frac{d\xi}{(q(s))^{\frac{1}{2}}} \frac{V^{1/5}}{s - \xi} ds \]

Now, however, in general, \( G(z) \) does not decay as \( z \to \infty \). Indeed

(162.2) \[ G(z) = \left( \frac{3^k + \ldots}{2\pi i} \right) \left( -\frac{1}{3} \right)^k \sum \int ds \frac{d\xi}{(q(s))^{\frac{1}{2}}} \frac{V^{1/5}}{s - \xi} \left( 1 + \frac{s}{3} + \cdots + \frac{s^{k-1}}{3^{k-1}} \right) \]

and to ensure that \( G(z) \to 0 \) as \( z \to \infty \), the

\[ k \text{ moment conditions} \]

(162.3) \[ \sum \int \frac{V^{1/5}}{q(s)^{\frac{1}{2}}} s^j ds = 0, \quad j = 0, \ldots, k-1. \]

must be satisfied.

This gives \( k \) conditions on the \( 2k \) endpoints \( a_1, b_1, \ldots, a_k, b_k \). Furthermore, from (158.2),

(162.4) \[ G(y) = -\frac{1}{i\pi y} \int \frac{4(\xi)}{y} d\xi + O(\frac{1}{y^2}) \]

\[ = -\frac{1}{i\pi y} + O(\frac{1}{y^2}) \]

as \( \int 4 dx = 1 \).
This leads to the additional condition from (162.2)

\[(163.1) \quad \frac{i}{2\pi} \int \frac{V^{1/3} s^k}{\Sigma (q(s))^{1/2}} ds = 1\]

Thus we still require \( k-1 \) relations to determine the endpoints of \( \Sigma \). These are obtained via the relation

\[(163.2) \quad \frac{d}{dx} \left[ \int [2 \left( \log |x-y| + u(y) \right) dy + V(x)] \right] = -2\pi i H(x) \]

In order that (iii) is satisfied with the same constant \( d \) in all the \( k \) intervals \((a_i, b_i), 1 \leq i \leq k\), we must have

\[\frac{a_{i+1} - b_i}{a_i - b_i} = \frac{a_{i+1} - b_i}{a_i - b_i}\]

Equations (163.3) provide the remaining equations, in addition to (163.3) and (163.1), for \((a_1, b_1, \ldots, a_k, b_k)\).

In addition to (162.3) and (163.1) and (163.3) we have
Ke side conditions

\begin{equation}
\text{Re} \mathcal{G}_K(x) = 4 \chi_1 \geq 0 \quad \text{and} \quad \{\text{Re} \mathcal{G}_K(x) > 0\} = \Sigma
\end{equation}

\text{i.e., supp} \{4 \chi_1 dx\} = \Sigma.

\text{and (ii)},

\begin{equation}
2 \int \log |x-y|^{-1} 4(y) dy + V(x) \geq \epsilon \quad \forall x \in \Sigma.
\end{equation}

As

\[\epsilon = 2 \int \log |b_i - y|^{-1} 4(y) dy + V(b_i), \quad i = 1, \ldots, k - 1\]

we see that (164.2) can be written, using (163.2)

\begin{equation}
\int \left(4(y) - \frac{V'(y)}{2 \pi} \right) dy \leq 0, \quad b_i - x \leq a_i - x, \quad i = 1, \ldots, k - 1.
\end{equation}

\begin{equation}
\int \left(4(y) - \frac{V'(y)}{2 \pi} \right) dy \geq 0, \quad x < a,
\end{equation}

\begin{equation}
\int \left(4(y) - \frac{V'(y)}{2 \pi} \right) dy \leq 0, \quad x > b_k.
\end{equation}

To summarize, the above calculations show that

if \(\tilde{f}^e = \mu_1 \chi_1 dx\), where \(\mu_1 \chi_1\) is a cont. func. of

\[\Sigma = \bigcup_{i=1}^k (a_i, b_i)\]

compact support, \(\lambda\), then conditions \(163.3, 163.1, 163.3, 164.1\) (and \(164.3\))
must be satisfied. Conversely, suppose that $\Sigma$

is a union of intervals $\bigcup_{i=1}^k (a_i, b_i)$, and define $G(3)$

and suppose that $G(3)$ is continuous down to the axis from $a_i$.

by (162.1) Then if the pair $(\Sigma, \chi(x) = \Re G(x))$

satisfies (162.3)(16.3)(16.3) (164.1) and (164.3), then one

can show that $\chi(x)dx$ is the density measure for $\chi(x)$. (Note

that as $\chi(x) = \frac{1}{2}(G(x) + \Re G(x))$, $\chi(x)$ is continuous by assumption.)

Indeed, from (162.1) we see that $G(3) = \Theta(\frac{1}{b})$ as $\frac{1}{b} \to 0$

and hence, we are assuming that $G(3)$ is continuous down

to the real axis from $a_i$ and $\chi(x)$, see below, by a simple

application of Cauchy's Theorem, we find that

$$G(3) = \frac{1}{2\pi i} \int_{a_i}^{b_i} \frac{G(x) - G_{+}(x)}{x - \zeta} \, dx.$$ 

However, as $G_{+}(3) = \Omega(3)$ for $3 \in \Sigma$, an

$$\overline{G(3)} = \overline{G(3)}$$ for $3 \in \Sigma \setminus \Sigma$, we see that

$$G(3) + \overline{G(3)} = 0 \quad \forall \zeta \in \Sigma \setminus \Sigma$$

and hence

$$G_{+}(x) + G_{-}(x) = 0 \quad \forall x \in \mathbb{R}.$$
and in particular, for \( x \in b_i \) or \( x \in a_i \), where

\[ G(z) \text{ is analytic, so } 2 \pi i = G_+(x) + \overline{G_+(x)} = G_+(x) + G_-(x) = 0. \]

Similarly, \( 4(x) = 0 \) for \( x \) in the gaps, i.e., \( x \in (b_i, a_{i+1}) \), \( i = 1, \ldots, k - 1. \)

(Note: (164.1) Could be replaced with the weaker assertion: \( (164.1', \quad \sum_{i \in \mathbb{Z}} 1) < \infty \). For by the above calculation, we see that \( 4(x) = 0 \) on \( \mathbb{R} \setminus \Sigma \), then (164.1) implies (164.1').

Thus, \[ G(z) = \frac{1}{2\pi i} \int \frac{G_+(s) - G_-(s)}{s - z} \, ds \]

\[ = \frac{1}{2\pi i} \sum \int_{C_i} \frac{G_+(s) + \overline{G_+(s)}}{s - z} \, ds \]

\[ = \int_{\Sigma} 4(s) \, ds \]

As \( m(159.1) \) we learn that \( G_{\pm}(x) = \pm 4(x) + i 4(x) \),

which implies that \( G_+ + G_- = 2i 4(x) \). But then (162.1),

for \( x \in \Sigma \), \[ G_+(x) + G_-(x) = \frac{i}{\pi} V(x) \quad \text{Hence} \]

\[-2\pi i 4(x) + V'(x) = 0, \quad x \in \Sigma \]

This implies as before that \( 2 \int (x, x-y_1, 4(x)dy + V(x)) \)

is constant on \( \Sigma = 44(x) = c \), and only (163.3) it must
\[ (16.7) \]

Let the same constant, say \( k \), in each interval \((a_i, b_i)\).

Finally, as \((16.4.3)\) is satisfied by assumption, we conclude that \((16.4.2)\) is true. It then follows by Theorem 156.1 that \( y(x)dx = \mu G(x)dx \), which is a probability measure by 

\((16.4.1)\) and \((16.3.1)\), so the equilibrium measure for \( V(x) \).

**Important Remark:**

As noted above, \((16.1.3)\), \((16.5.1)\) and \((16.3.3)\) give 2k conditions for the 2k points \( a_1, a_2, \ldots, a_k \). This is true for any \( k \geq 1 \), and it may happen that we obtain solutions of these equations for many values of \( k \).

However, amongst all these solutions only one of them can solve the side conditions \((16.4.1)\) and \((16.4.3)\), and that one is the desired solution (see below)
Exercise: Show that if \( V \subseteq C^2(\mathbb{R}) \), then \( G(z) \) defined by (162.1) is continuous up to the boundary from \( C_+ \) and \( C_- \).