Said differently, we have produced a solution

\[ q(x, \theta) = \left( \frac{q_1(x, \theta), q_2(x, \theta)}{a(x)} \right), \quad \text{for } \theta > 0 \]

of (107.1) such that

\[ (109.1) \quad \left( U(x, \theta) e^{-ix\theta} \right) \rightarrow I = (0, 0) \quad \text{as } x \to -\infty \]

and is bounded as \( x \to 1 + \infty \).

Such solutions are called Real- Coifman (RC) solutions of (107.1).

Such a solution of (107.1), even in the general case \( q \in L^1(\mathbb{R}) \), if it existed, would be unique.

Indeed, suppose there were 2 such solutions, \( q \) and \( \tilde{q} \). Then as \( \theta \to (i3\pi + Q_i) = 0 \)

it follows that \( \det q(x, \theta) = \text{const} = c(2) \).

But as \( x \to -\infty \), \( \det q(x, \theta) = \det (q(x, \theta) e^{-ix\theta}) \to 1 \)

and no \( c(1) = 0 \). Hence, 7 matrix \( B = (9/3) \) such that

\[ \tilde{q}(x, \theta) = q(x, \theta) B. \]
\[ \tilde{u} = m e^{i3\pi} \quad 4 = m e^{i3\pi}, \quad \text{with} \]
\[ \tilde{m}, m \to 1 \quad \text{as} \quad x \to -\infty \]
and are bounded as \( x \to +\infty \).

Then
\[ \tilde{m} = m e^{i3\pi} B e^{-i3\pi} = m \begin{pmatrix} B_{11} & B_{12} e^{i3\pi} \\ B_{21} e^{-i3\pi} & B_{22} \end{pmatrix} \] (110.1)

Letting \( x \to -\infty \), we conclude that \( B_{12} = 0 \) and
\[ B_{11} = B_{22} = 1 \quad \text{as} \quad \tilde{m}, m \to 1, \quad \text{but then} \]

Letting \( x \to +\infty \), we conclude that \( B_{21} = 0 \). Hence \( B \to 1 \) and so \( 4 = \tilde{u} \).

We also have uniqueness for Beals-Loitman solutions in \( \Im z \leq 0 \).

(Note that such solutions do not exist for \( 3 \in \mathbb{R} \); we needed the exponential growth in (110.1) to conclude uniqueness.)
New there is a mantra in the theory of
integrale systems, and indeed in mathematics as
large, that if an object is singled out, or
uniquely specified, then it’s the right object to
consider. The converse of this statement is
probably even more true: if you cannot define
precisely what one means by something, then
you probably won’t get very far!

So one concentrates on these BC solutions.

It turns out (see [D Zhou, above]) that such
solutions $4(r, z)$

- exist for all $g < L'$, $3 \in \mathbb{R}$.
- are analytic in $C \backslash \mathbb{R}$ for any $x \in \mathbb{R}$.
- are continuous down to the axis,
\[ \Psi_+(x,3) = e^{i(x,3) \omega} \]

and satisfy
\[ \Psi(x,3) e^{-i \Delta x} \rightarrow I \text{ as } x \rightarrow \infty \text{ in } C \backslash \mathbb{R} \]

To higher order, \( e^{i \Delta x} \rightarrow 1 + i \frac{x}{\Delta} + O(\frac{1}{x^2}) \) and we have

\[ q(x) = -i (m(x,3) \Delta) \]

Hence
\[ \Psi_+(x,3) = \Psi(x,3) \psi(x,3) \]

for some matrix \( \psi(x,3) \), with \( \psi(x,3) = 1 \). Direct calculation (see [23]) shows that \( \psi(x,3) \psi^T \) to form

\[ \psi(x,3) = \begin{pmatrix} 1 - |r(x,3)|^2 & r(x,3) \\ -\overline{r(x,3)} & 1 \end{pmatrix} \]

where
\[ r = r(x,3) \] is "reflection coefficient"

\[ \| r \|_\omega < 1 \]

To summarize, we see that
\[ m = m(x,3) = \Psi(x,3) e^{-i \Delta x} \]

\[ \Psi_+(x,3) = \Psi(x,3) \psi(x,3) \]

\[ q(x) = -i (m(x,3) \Delta) \]

\[ \psi(x,3) = \begin{pmatrix} 1 - |r(x,3)|^2 & r(x,3) \\ -\overline{r(x,3)} & 1 \end{pmatrix} \]

\[ r = r(x,3) \]

\[ \| r \|_\omega < 1 \]
The classical solution of the normalized RH problem (113.1):

\[
\Sigma = \mathbb{R}, \quad v_x = e^{i3x} v e^{-i3x} = \begin{pmatrix} 1 - 1 & e^{i3x} \\ e^{-i3x} & 1 \end{pmatrix}
\]

In this way the scattering problem (107.1) gives rise to a RH problem. Scattering and inverse scattering theory consists of the study of the map

\[ q \rightarrow r = R(q) \]

and its inverse \( r \rightarrow q = R^{-1}(r) \).

\( R \) is constructed as follows from the scattering problem (107.1):

\[ q \rightarrow 4(x, 3; q) \rightarrow v_x(3) \rightarrow r = R(q), \quad \text{BC solution} \]

\( R^{-1} \) is constructed as follows from the RH problem \( (\Sigma, v_x) \):

\[ r \rightarrow v_x = m(x, 3; r) = \mathcal{I} + \frac{m_1(x; r)}{3} + \mathcal{O}\left(\frac{1}{3}\right), \quad 3 \rightarrow \infty \rightarrow -i\left(m_1(x; r)\right)_{12} = q = R(r). \]
At the technical level, one shows (see (D.77)) that $\mathcal{R}$ is a bi-Lipschitz (i.e., $\mathcal{R} \cdot \mathcal{R}^{-1}$ are Lipschitz) isomorphism from

$$H^{1,1} = \{ q \in L^2 : x q, q' \in L^2 \}$$

onto

$$H^{1,1} = H^{1,1} \cap \{ \| q \|_\infty < 1 \} = \{ q \in L^2 : x q', \| q \|_\infty < 1, \| x q \|_\infty < 1 \}$$

Moreover, if $q = q(t)$ solves NLS, $q(x, t=0) = q_0(x) \in H^{1,1}$, then $r(t) = R q(t+1)$ evolves simply (this is the analog for continuous spectrum of the fact that the pt. spectrum, in the periodic case, remains fixed in time)

$$r(t) = r(t, 3) = r(t, 3) e^{-i t \delta} , \delta \in \mathbb{R}$$

Thus we have the solution procedure for NLS:

$$q(t) = R^{-1} (e^{-i t \delta^2} R q(0))$$

The efficacy of the method to compute the long-time asymptotics of $q(t)$, rests on the fact
That we can control the solution of the oscillating RHP
\[ (\Sigma, \nu_{x,t} = \begin{pmatrix} 1 - (i)^{\theta} & r e^{i\theta} \\ -r e^{-i\theta} & 1 \end{pmatrix}) \]
for \( \theta = x_3 - t_3 \), as \( t \to \infty \), \( x \in \mathbb{R} \), using the non-commutative steepest-descent method for RHP's. Think of \( \mathbb{R} \) as a non-linear version of the Fourier transform \( \mathcal{F} \) and indeed, for small , \( t^3 = (i)^{\theta} \). Thus we have seen 4 sources for RHP's:

1) Integrable operators

2) Wiener-Hopf type problems: \( (1 - k)^2 = q \) on \( L^2(\mathbb{R}^+) \), \( k(x, y) = k(x-y) \).

3) Scattering problems

4) "Out of the blue" e.g. RHP for orthogonal polynomials \( (FkI)^7 \).

Problems from [Fokas, Its, Novokshenov, Kap107].

The Painlevé equations are also described by

RHP's: such RHP's arise by considering lax-pairs.
As in § above. Recall that an equation of the form

\[ y'' = F(x, y, y') \]

\[ y(x_0) = a, \quad y'(x_0) = b \]

(F meromorphic in \( x \), continuous and \( y, y' \))

has the Painlevé property if the following is true: The only singularities in the complex plane of the solutions \( y = y(x; a, b) \) of (116.1) that are allowed to move with \( a, b \), are poles. Any essential singularities, branch points, must remain fixed as \( a, b \) vary. Up to change of variables, there are precisely 6 new Painlevé transcendents.

**Exercise:** Consider:

(i) \( y'' = y \)

(ii) \( y'' = -\frac{1}{4} y^3 \)

(iii) \( y'' = 2y y' \)

Show that (i) (iii) have the Painlevé property, but (iii) does not.

The general form of the Painlevé III equation is

\[ u'' = x u + 2u^3 + v , \quad v = \text{const} \]
We will illustrate the situation for P_II with n = 0

\[ u'' = x u + 2 u^3 \]

Case \[ Fok, Its, Aka, Nov \]

for \( n \to 0 \) case

In this paper, we will show how to construct the associated Riemann \( \frac{1}{2} \) use it

prove the Painlevé prop. for (117.1). Similar considerations apply for all 6 Painlevé systems (see Ito et al.). History:

Hirota, Segur, Sato-Miwa-Jimbo, Fuchssteiner-Newell,

The key fact is that P_II is equivalent to

\textit{The compatibility of the following Lax Pair:}

\[ \begin{align*}
\frac{\partial u}{\partial \xi} &= \begin{pmatrix}
4i \xi - i x - 2 i u \\
-4i \xi - 2 w
\end{pmatrix} \quad y = L_4 \\
\frac{\partial u}{\partial x} &= \begin{pmatrix}
-i \xi & i u \\
-i u & i \xi
\end{pmatrix} \quad y = P_4.
\end{align*} \]

Here \( u = u(x, z), \quad w = w(x, z) \). It is a simple exercise to verify that the compatibility of 2nd derivatives

\[ \frac{\partial^2 y}{\partial \xi \partial x} = \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x^2} \]
In equivalent to the relations

\[ u = u_x, \quad -u_x + xu + xu^3 = 0 \]

and to

\[ u_{xx} = xu + 2u^3 \]

(118.2) Note that (117.2) (117.3) \( \Rightarrow \ L_x = P_3 + [P, L] = [\partial_x P, \partial_3 - L] = 0 \)

In analogy with NLS, the analysis of PT process by considering distinguished solutions of the spectral problem

(118.3)

\[ \frac{\partial \psi}{\partial \phi} = L \psi \]

(Note that in contrast with NLS, \( \phi \) now plays the role of \( t \) and \( \psi \) plays the role of \( x \).)

Write

\[ L = 3^2 A_2 + 5 A_1 + \Phi_0 \]

where

(118.4)

\[ \begin{align*}
A_2 &= -u \otimes \delta_3 \\
A_1 &= -4u \otimes \delta_2 \\
A_0 &= -(i \delta_x + 2i \delta_1) \delta_3 - 2u \delta_1
\end{align*} \]
where \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

We think of \( z \) as fixed and \( u(x) \), \( u(x) = u(x) \) is fixed and seek \( y = y(z, x) \) as a function of \( z \).

Now there is a general theory for solutions of such systems (11.8.3), (11.8.4) (see e.g. W. Wasow, *Asymptotic Expansions for ODEs*, Interscience, NY, 1965, Chap IV).

The pt \( z = \infty \) is a so-called irregular singular pt. for (11.8.3) which is a particular example of an equation of the form

\[
\frac{dy}{dz} = g(z) y \quad \text{where} \quad g(z) = \frac{A(z)}{z^N}, \quad z \in \mathbb{C} \setminus \{0\},
\]

Thm. (Wasow p. 60 Thm 12.3; see also Thm 12.2, p. 58)

Let \( S \) be an open sector in the \( z \) plane with vertex at the origin and a positive opening angle not exceeding \( \pi/(d+1) \). Let \( A(z) \) be an \( n \times n \) matrix.
function holomorphic in \( S \) for \( |\beta| > 3 \alpha > 0 \) and admitting
an asymptotic series

\[(120.1) \quad A(\beta) = A_0 + \frac{A_1}{\beta} + \frac{A_c}{\beta^c} + \ldots, \quad \beta \to \infty, \beta \in S\]

Assume in addition that the eigenvalues of \( A_0 \) are

distinct. Then (119.1) admits a fundamental matrix

solution of the form

\[(120.2) \quad \gamma(\beta) = \gamma(\beta) \beta D \in \mathcal{A}(\beta)\]

Here \( \mathcal{A}(\beta) \) is a diagonal matrix whose diagonal
elements are polynomials of degree \( q+1 \). The leading
term of \( \mathcal{A}(\beta) \) is

\[(120.3) \quad \frac{\beta^{q+1}}{q+1} \text{ diag } (\lambda_1, \ldots, \lambda_n)\]

\( D \) is a constant diagonal matrix, and the matrix

\( \gamma(\beta) \) has an asymptotic expansion as \( \beta \to \infty, \beta \in S \),