Lecture: The RHP's for MKdV and Painlevé II were presented out of the blue and without a derivation or a proof. Here is an example (see P.D. OP's and Random Matrices) that one can verify directly.

Let \( q(x) = e^{i\lambda x} \), \( w(x) \equiv 0 \), be a measure with finite moments \( \mu_k \):

\[
\int |x|^k w(x) dx < \infty \quad k = 0, 1, 2, \ldots
\]

and let \( (P_k(x)) \) be the orthonormal polynomials associated with \( q(x) \) obtained by orthogonalizing \( 1, x, x^2, \ldots \) with \( q(x) \) (Gram-Schmidt procedure). We have

\[
P_k(x) = \delta_k x^k + \cdots \quad \delta_k > 0, \quad k \geq 0,
\]

and

\[
\int P_k(x) P_l(x) w(x) dx = \delta_{kl} \quad \delta_{kl} \geq 0
\]

Exercise: let \( P_k(x) \) and \( \phi(x) \) unique, \( k \geq 0 \). Let \( \Phi_k(x) = \frac{1}{\delta_k} P_k(x) = x^k + \cdots \); \( \Phi_k \) is the monic orthog.
Orthogonal polynomials (OP's) are fundamental objects in analysis and their asymptotics as $k \to \infty$ are of great interest. For example, The proof of universality for invariant random matrix ensembles, reduces to the evaluation of the asymptotics of OP's. Some of the classical OP's are the following:

- $a_n(x) = e^{-x} \, dx \quad \Rightarrow \quad P_n(x) = c_n H_n(x) = c_n x^n \cdot \text{Hermite poly,}\quad c_n \text{ const.}$

- $a_n(x) = (1-x) x^\alpha (1+x)^\beta \, dx, \quad -1 < x < 1 \quad \Rightarrow \quad \text{Jacobi polynomials}$

- $a_n(x) = e^{-x} \, dx, \quad x > 0 \quad \Rightarrow \quad \text{Laguerre polynomials.}$

Each of the classical poly's has an integral representation, e.g. (see Szegö "Orthog. Polynomials")

$$H_n(x) = \frac{n!}{2^n} \int_{-\infty}^{\infty} e^{2xw - w^2} \, dw, \quad \text{and converges 0}.$$
As noted before, the asymptotics of \( H_n(x) \) as \( n \to \infty \) (and also as \( x(\to \infty) \) can then be inferred from (20.1) using the classical steepest descent method. However, for \( e^{-x^2} \text{d}x \), say, no such integral formulae are known. However, we do have a RHP!

Let \( w(x) \text{d}x \) have finite moments as above. Let \( \Sigma = \mathbb{R} \) oriented from \( -\infty \to +\infty \) and let \( \psi(z) = \begin{pmatrix} i \omega(z) \\ 0 \end{pmatrix}, \quad z \in \mathbb{R}. \) Fix \( n > 0 \). We seek a \( 2 \times 2 \) matrix-valued function \( Y = Y^{(n)}(z) \) such that

\[
\left\{
\begin{array}{l}
Y = Y^{(n)}(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma \\
Y_{+}(z) = Y_{-}(z) \begin{pmatrix} 1 & \omega(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R},
\end{array}
\right.
\]

when \( Y_{\pm}(z) = \lim_{\varepsilon \to 0} Y(z \pm i \varepsilon), \quad z \in \mathbb{R} \)

\[
Y(z) \left( \begin{array}{cc} e^{-n} & 0 \\ 0 & 3^n \end{array} \right) \to I \quad \text{as} \quad z \to \infty.
\]
The proofs and calculations that follow are somewhat formal, fully rigorous proof later.

Claim: \( y \) exists, it is unique.

Indeed if \( y \) exists, then \( \det y(3) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \) and \( \det y(3) \) is analytic across \( \Sigma = \mathbb{R} \), and hence entire.

But as \( z \to 0 \), \( \det y(z) = \det \left( y(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \to 1, \)

as \( z \to 0 \), \( \det y(z) = 1 \), by Liouville. Now suppose \( y'(3) \) is another solution of (21.1). Set \( R(\beta) = y(\beta) / y'(\beta) \).

Note that as \( \det y(z) = 1 \), \( y'(\beta) \neq 0 \) and is analytic in \( \mathbb{C} \setminus \mathbb{R} \). Thus

- \( R(\beta) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \)

- \( R_{+}(\beta) = y_{+}(\beta) (y_{+}(\beta)^{-1} = \left( y_{-}(\beta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = R_{-}(\beta) \),
and so $R(\beta)$ is analytic across $\Sigma$ as $n \to \infty$.

Hence $R(\beta) = I$, again by Liouville \( \Rightarrow \hat{\gamma}(\beta) = \gamma(\beta) \).

**Remark:** This proof, simple as it is, is prototypical in Riemann theory.

Write

\[(23.1) \quad \gamma(\beta) = \begin{pmatrix} \gamma_{11}(\beta) & \gamma_{12}(\beta) \\ \gamma_{21}(\beta) & \gamma_{22}(\beta) \end{pmatrix} = \left( I + O(\frac{1}{\beta}) \right) \begin{pmatrix} 3^n & 0 \\ 0 & 3^{-n} \end{pmatrix} \]

Suppose $n > 1$. The first row of $\gamma_t = \gamma(\frac{1}{\beta}, i)$

reads as follows

\[(23.2) \quad \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}_+ = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}_- \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \]

In particular

\[(\gamma_{11})_+ = (\gamma_{11})_- \]

\( \Rightarrow \) $\gamma_{11}(\beta)$ is entire. But from (23.1)

\[\gamma_{11}(\beta) = 3^n \left( 1 + O(\frac{1}{\beta}) \right) \quad \text{as} \quad \beta \to \infty \]

Hence

\[\gamma_{11}(\beta) = \text{polynomial} = 3^n + \ldots \]
Now

\[(Y_{12})^+ = (Y_{12})^- + \omega \cdot Y_{11}(3)\].

Set

\[(24.1) \quad h(3) = \int_\mathbb{R} \frac{w(x) \cdot Y_{11}(x)}{x - 3 - \frac{\varepsilon}{x - 3 + \frac{i\varepsilon}{x}}} \, dx, \quad \varepsilon \in \mathbb{C} \setminus \mathbb{R}.

Note that as \(w \cdot r_2n\) has finite moments, \((h_2)\) integrated \(\mathbb{F}\).

Then \(h(3)\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\) and for \(3 \in \mathbb{R}\),

\[h_+(3) - h_-(3) = \frac{\varepsilon}{2\pi} \int_\mathbb{R} \frac{1}{x - 3 - \varepsilon - \frac{1}{x - 3 + \varepsilon}} w(x) \cdot Y_{11}(x) \, dx\]

\[= \frac{\varepsilon}{2\pi} \int_{\varepsilon \to 0} \frac{\omega(x) \cdot Y_{11}(x)}{(x - 3)^2 + \varepsilon^2} \, dx\]

\[= w(3) \cdot Y_{11}(3).

Thus

\[(Y_{12} - h)^+ = (Y_{12} - h)^-\]

and \(Y_{12} - h\) is entire. But from (23.1)

\[Y_{12}(3) = O(3^{-n-1}) \quad \text{as} \ 3 \to 0 \quad \text{and} \quad h(3) = O(\frac{1}{3})\]

as \(3 \to 0\); hence \((Y_{12} - h)/(3) \to 0 \quad \text{as} \ 3 \to 0\). Thus, again (24.2) by Liouville,

\[Y_{12}(3) = h(3) = \int_\mathbb{R} \frac{w(x) \cdot Y_{11}(x)}{x - 3} \, dx.

\]
Expanding (24.2) as $z \to \infty$, we have

$$
Y_n(3) = -\frac{1}{2\pi i} \oint \frac{\Psi_{\pi}(x) \left( \frac{1}{x} + \frac{\pi}{\delta} + \cdots + \frac{x^{n-1}}{3^n} + \frac{x^n}{3^{n+1}} + \cdots \right) dx}{x^i}
$$

As $Y_n(3) = O(3^{-n-1})$, as $n \to \infty$, we must have

$$
\int Y_n(x) x^i \, \Psi_{\pi} \, dx = 0, \quad 0 \leq i \leq n-1.
$$

It follows that necessarily

$$Y_n(x) = \Pi_n(x) = n^{th \ monic \ op \ for \ \Psi_{\pi} \ \delta x}.$$

Thus

$$
(Y_{n+1/2})_n \leq \left( \frac{\Pi_n(x)}{\Pi_n(x)} \frac{\Psi_{\pi}(x) \, dx}{2\pi i} \right)
$$

Similarly (exercise) we find that

$$
(Y_{n+1/2})_n = \left( -\frac{2\pi i \delta_n}{\Pi_{n-1}(x)} \frac{\Pi_{n-1}(x) \, dx}{2\pi i} \frac{\Psi_{\pi}(x) \, dx}{x-3} \right)
$$

Conversely, if $\{Y_{n+1/2}\}_{1 \leq i \leq 2}$ are given as per (25.3)/(25.4),

New $Y = \left( \begin{array}{c} \Psi_{\pi} \\ \Psi_{\pi} \end{array} \right)$ satisfies (24.1). We have

proved the following result:
Proposition 26.1 (Fokas-Its-Kitaev). For $n \neq 0$,

\[
Y(n) = \pi \int_{\mathbb{R}} w(x) \pi_n(x) \frac{dx}{2\pi i} \cdot \int_{\mathbb{R}} -2\pi i \delta_{n-1} \pi_{n-1}(x) w(x) \frac{dx}{2\pi i}.
\]

is the unique solution of the RHP $(\Sigma = \mathbb{R}, \nu = (0, 1))$ nor-malized so that

\[(26.2) \quad Y(n)(\delta - n \delta) \rightarrow I \quad \text{as} \quad z \rightarrow \infty .
\]

(For $n = 0$, $Y = \left( \begin{array}{c} 1 \\
\int_{\mathbb{R}} w(x) \frac{dx}{x-\delta} \end{array} \right)$)

Applying the non-linear steepest descent method to $(I, \nu)$ as $n \to \infty$, we can, in particular deduce

the asymptotics of $\pi_n(x)$, for very general weights $w(x) \, dx$.

Orthogonal polynomials famously satisfy a 3-term recurrence relation

\[(26.3) \quad b_{n-1} p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x) = x \, p_n(x), \quad n \geq 0,
\]
\( a_n \in \mathbb{R}, b_n \to a_{n0} \text{ for } n \to \infty (b_n \equiv 0) \). As \( n \to \infty \), one is interested not only in \( \ell \)-asymptotics of \( \pi_n(x) \) but also in the asymptotics of \( a_n, b_n \) and the normalising constants \( \delta_n \). These 3 quantities can be read off directly from the RHP: we have

\[
\delta_{n-1}^2 = -\frac{1}{2\pi i} (y_i^{(n)})_2^1
\]

(27.11)

\[
\begin{align*}
an &= (y_i^{(n)})_1^1 - (y_i^{(n+1)})_1^1 \\
b_{n-1} &= (y_i^{(n)})_1^1 (y_i^{(n)})_2^1
\end{align*}
\]

where

\[
y = y^{(m)} = \left( I + \frac{1}{3} y_i^{(m)} + O(\frac{1}{3^2}) \right) \left( \begin{array}{cc} 3^n & 0 \\ 0 & 3^{-n} \end{array} \right), \quad 3^{-\infty},
\]

so the RHP captures all the quantities of basic interest for OP's.
Up till now, we have presented RHP's as a tool to evaluate physical and mathematical quantities asymptotically as some associated parameter goes to infinity. But RHP's are also very useful in analytical and algebraic contexts. We will develop these themes later, but just to illustrate how things work, we will now show how the RHP for the OP's can be used to derive the difference equation (26.3).

The jump matrix $\Sigma$ of a RHP $(\Sigma, \nu)$ is independent of a parameter in the problem. Thus variations in that parameter give rise to differential/difference equations.
To apply this method to the RHP for OP's, observe that $\nu$ only appears in the asymptotics

\[
\chi \left( \frac{3^n}{5^n} \right) \to I, \quad \text{but not in } \nu = (0, 1).
\]

Let $\chi(n)$, $\chi^{-1}(n)$ be the solutions of the OP RHP (21.1). Let $T = \chi^{-1}(n) \chi(n-1)$. Then

\[
T(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}.
\]

\[
T(z) = \left( \chi^{-1}(n) \right) \left( \chi(n-1) \right)^{-1}
\]

\[
= \left( \chi^{-1}(n) \right) \left( \chi(n-1) \right)^{-1}
\]

\[
= \left( \chi^{-1}(n) \right) \left( \chi(n-1) \right)^{-1}
\]

\[
= T(z), \quad z \in \mathbb{R},
\]

and so $T(z)$ is entire as before.

Let $\beta = 3 - \alpha$

\[
T(z) = \chi^{-1}(n) \chi(n-1) \chi(n-2)
\]

\[
= \chi^{-1}(n) \chi(n-1) \chi(n-2)
\]

\[
= T(z) + O(1)
\]

as $z \to \infty$.

Thus by Liouville

\[
T(z) = (3, 0) + A_n
\]

for some constant matrix $A_n$. 
In other words, we have shown that

\[ \gamma^{(n-1)}(\beta) = ((0, 0) + A_n) \gamma^{(n)}(\beta) \]

for some constant matrix \( A_n \). Evaluating \( A_n \), we then obtain, in particular, the recurrence relation (26.3) (exercise).

Now from an analytic point of view, what kind of problem is a RHP? As we will see, the Cauchy operator (cf. (24.11))

\[ C h(\beta) = \int_{\gamma} \frac{h(s)}{s - \beta} ds, \quad \beta \in \mathbb{C} \setminus \Sigma \]

will play a central role. We saw above that for \( \Sigma = \mathbb{R} \),

\[ C^+ h(\beta) - (-C^- h)(\beta) = h(\beta) \quad \beta \in \mathbb{R} \]

(E.0.1)

\[ C^+ - C^- = 2 = id \]

The same is true for the Cauchy operator on very general
oriented contours $\Sigma$

Let a jump matrix $\mathcal{J}$ on $\Sigma$ be given and suppose $m(\beta)$ denotes the normalized NTH $(\Sigma, \mathcal{J})$.

* $m(\beta)$ is analytic in $\mathbb{C} \setminus \Sigma$

* $m_+(\beta) = m_-(\beta) \mathcal{J}(\beta)$, $\beta \in \Sigma$

* $m(\beta) \to I$ as $\beta \to \infty$

How do we compute $m(\beta)$?

**Method:** Let $\mu(\beta) = I + \Phi(\beta) \mathcal{J}(\beta)$, where $\Phi(\beta) \to 0$ as $\beta \to \infty$ in $\Sigma$.

in some specific sense and suppose $\mu$ solves the singular integral equation

$$(3.1.1) \quad (1 - \mathcal{C}_\beta) \mu = \mathcal{I}$$

Here $I$ on the RHS denotes the function which is constant $I$ a.e. in $\Sigma$.

$$C_{\mathcal{J}, \mu}(\beta) = \mathcal{C}_{\mathcal{J}, \mu}(\beta) = \mu^\dagger \mathcal{C}_{\mathcal{J}, \mu}(\beta-1) \mathcal{C}_{\mathcal{J}, \mu}(\beta)$$
Alternatively, writing \( m = I + \phi \), we have

\[
(1 - C_{\nu}) (I + \phi) = I
\]

Thus

\[
(1 - C_{\nu}) \phi = C_{\nu} I = C_{\nu} (\nu - I)
\]

Now set

\[
m_{(b)} = I + C (\mu (\nu - I)) (b), \quad b \in \mathbb{C} \setminus \Sigma
\]

\[
= I + \int_{b}^{\infty} m_{(b)} (s) (s) \, ds
\]

Clearly \( m_{(b)} \) is analytic in \( \mathbb{C} \setminus \Sigma \), and \( m_{(b)} \to I \) as \( b \to \infty \).

Then on \( \Sigma \)

\[
m_{+} = I + C (\nu - I)
\]

\[
= I + C_{\nu} (\nu - I) + \mu (\nu - I), \quad \text{by (30.1)}
\]

\[
= I + C_{\nu} \mu + \mu (\nu - I)
\]

\[
= \mu
\]

Similarly

\[
m_{-} = I + C_{\nu} (\nu - I)
\]

\[
= \mu
\]

Hence \( m_{+} = \mu v = m_{-} \nu \quad \text{and} \quad \mu \nu \quad m_{(b)} \) solves the normalized RH problem \((\Sigma, \nu^{-})\).
In this way the RHP reduces to the study of the singular integral equation (31.1).

Questions:

- In which space do we try to solve the equation? At the very least the operators \( C^\pm \) should be based in this space.
- Is the solution unique in this space?

- Does the operator \( L_C \) have any special structure? In particular, is it Fredholm?

The RHP is studied in a variety of spaces (see, e.g., Clancy and Goldberg). For example,

- \( L^p(\Sigma, \mu_1) \) spaces, \( 1 < p < \infty \)
- \( W^{k,p}(\Sigma, \mu_1) \) spaces, \( 1 < p < \infty \), \( k = 0, 1, 2, \ldots \)
- Spaces of Hölder continuous functions

Most commonly we study the RHP in \( L^1 \), \( 1 < p < \infty \).

\( L^1 \) is the most important case. We will only consider the RHP in \( L^p \), \( 1 < p < \infty \).

Question: What conditions do we need to impose on \( \Sigma \) to obtain a viable \( L^p \) theory?

Firstly, we need a measure on \( \Sigma \) in order to
create a measure space on $\Sigma$. Consider first the case where

where $\Sigma$ is a simple continuous curve in $\mathbb{C}$ and

$$\Sigma = \{ \varphi(t) : a \leq t \leq b \}$$

where $\varphi(t)$ is continuous

when $\varphi(t) = \varphi(t') = \pm \infty$ unless, in the case that $\Sigma$ is a loop, and then $\varphi(a) = \varphi(b)$

We allow for the possibility that $\varphi(a)$ or $\varphi(b) = \pm \infty$.

The ordinary $a \rightarrow b$ pusher forward to a natural orientation on $\Sigma$. The minimal, standard way to place a measure on $\Sigma$ is to require that $\Sigma$ be locally rectifiable, i.e., if $3_0, 3_n$ are 2 pts on $\Sigma$, and $3_0, 3_1, \ldots, 3_n$ is any partition of $[3_0, 3_n]$, (3.11. succeeds) $3_i$ in the ordering on $\Sigma$, etc.)
Let \( L = \sup \sum_{i=0}^{n} (3i - 3i + 1) \leq \infty \)

All partitions \( \{3i\} \)

\( L \) is the arc-length of \( \Sigma \) from \( 30 \) to \( 3n \). For any interval \( (a, b) \) on \( \Sigma \), we define

\[
\mu((a, b)) = \text{arc-length } a \to b
\]

Now the sets \( \{(a, b) : a < b \text{ on } \Sigma\} \) form a semi-algebra (see Boyden: sets of type 2 such sets intersect again a set of the same type and the complement of \( (a, b) \) is a disjoint union of such sets) and hence \( \mu \) can be extended to a complete measure on a sigma algebra \( \mathcal{A} \) containing the Jordan sets. The restriction of the