We now show how the general theorem 174.1 can be used to solve a problem in radiative equilibrium.

(cf. H. Bynum & H. McKean, Fourier Series and Integrals, Milne's equation §3.6, pp 176-184):

"Think of the star as no big that its curvature can be neglected and introduce as coordinates, the depth 0 ≤ x ≤ 1 into the stellar interior and the inclination 0 ≤ θ ≤ π to the downward direction, as follows:

The distribution of radiation as regards depth and inclination is described by a "radiation density" e = e(x, θ)"
so that the amount of radiation in a slab \( \alpha \leq x \leq \beta \)

crossing at inclination \( \alpha \leq \theta \leq \beta \) is

\[
\int_{\alpha}^{\beta} \int_{\alpha}^{\beta} e(x, \theta) \sin \theta \, d\theta 
\]

The equilibrium is produced and maintained by streaming
(at speed \( v \), say) and by scattering, and a detailed

balance of these two mechanisms leads to (see ref to Hopf in
D'Arcy)

the following law:

\[
\cos \theta \frac{\partial}{\partial x}(e(x, \theta) + \delta(x, \theta)) = \frac{1}{2} \int_{0}^{\pi} e(x, \theta) \sin \theta \, d\theta
\]

Milne's problem is to compute the angular distribution

of radiation at the stellar surface

\[
I(\theta) = \frac{e(0, \theta)}{\frac{1}{2} \int_{0}^{\pi} e(0, \theta) \sin \theta \, d\theta}
\]

under the condition that no radiation is coming in

from the sky \( e(0, \theta) = 0 \) for \( 0 \leq \theta \leq \frac{\pi}{2} \), the so-called
"law of darkening". To do this, one introduces

the radiation intensity

\[ I(x) = \frac{1}{2} \int_0^\pi e(x, \psi) \sin \psi \, d\psi. \]

and solves

\[ \frac{\partial e}{\partial x} + \sec \psi \, e = e^{-x \sec \psi} \frac{\partial}{\partial x} (e^{x \sec \psi} \, e) = \sec \psi \, e \]

for \( 0 < \psi_1 \) and \( \psi > \frac{\pi}{2} \), separately, which leads to (the term at \( \psi = \infty \) is absent)

\[
\begin{cases}
  e(x, \psi) = \sec \psi \int_0^x e(y-x) \sec \psi \, f(y) \, dy, & 0 < \psi < \frac{\pi}{2} \\
  = -\sec \psi \int_x^\infty e(y-x) \sec \psi \, f(y) \, dy, & \frac{\pi}{2} < \psi < \pi
\end{cases}
\]

(4.0.1)

We now obtain:

\[
I(x) = \frac{1}{2} \int_0^\pi e(x, \psi) \sin \psi \, d\psi + \frac{i}{2} \int_{\pi}^{2\pi} e(x, \psi) \sin \psi \, d\psi
\]

\[
= \frac{1}{2} \int_0^{\pi} \sec \psi \left( \int_0^\pi e(y-x) \sec \psi \, f(y) \, dy \right) \sin \psi \, d\psi
\]

\[
+ \frac{i}{2} \int_{\pi}^{2\pi} (-\sec \psi \left( \int_x^\infty e(y-x) \sec \psi \, f(y) \, dy \right) \sin \psi \, d\psi.
\]
\[ = \frac{1}{2} \int_0^{\pi/2} \tan \theta \left( \int_0^\infty e^{(\gamma-x)\tan \theta} f(\gamma) \, d\gamma \right) \, d\theta \]

\[ = \frac{1}{2} \int_0^{\pi/2} \tan \theta \left( \int_0^\infty e^{-(\gamma-x)\sec \theta} f(\gamma) \, d\gamma \right) \, d\theta \]

\[ = \int_0^\infty h(x-y) f(y) \, dy, \quad x > 0 \]

where

\[ h(x) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\gamma \tan \theta} \, d\theta \]

Thus we are led to solving the following

Wiener - Hopf problem

\[ f(x) = \int_0^\infty k(x-y) f(y) \, dy, \quad x > 0 \]

with \( k \) as above. Equation (191.2) is known as

Milne's equation:

We want to show that (191.2) has non-trivial solutions, \( f(x) \neq 0 \) if we want to show that \( f \)
homogeneous Wiener-Hopf equations have a solution.

Now clearly

\[
|k(x)| \leq C e^{-\frac{1}{|x|}} \quad \text{as} \quad |x| \to \infty
\]

and \( k(x) \) has a logarithmic singularity at \( x=0 \). So in particular \( k(x) \in L^1(\mathbb{R}) \) and we can think of \( k(x) \) as \( L^2(\mathbb{R}_+) \).

By the calculations in Lectures 6 & 7, (19.1.2) is equivalent to a \( \mathbb{R}+\mathbb{P}_2 \) of the form

\[
(\text{see 9.1.5})
\]

\[
m_+ = m_- \cdot \mathcal{U} + H(3)
\]

Here

\[
m_+ = F(3) = \int_\mathbb{R} e^{ix} \cdot F(x) \frac{dx}{\sqrt{2\pi}}
\]

\[
H(3) = \psi'(3) \int_\mathbb{R} e^{ix} \cdot q(x) \frac{dx}{\sqrt{2\pi}} = 0 \quad \text{as} \quad q(x) = 0
\]

(19.1.2 is homogeneous)

\[
\psi(3) = (1 - K(3))^{-1}, \quad \text{provided} \quad k(3) \neq 1.
\]

Where

\[
K(3) = \int e^{ix} \cdot h(x) \frac{dx}{\sqrt{2\pi}}
\]

Thus we seek a solution \( m_+ \neq 0 \) to

\[
(19.2.3)
\]

\[
m_+ = m_- \cdot \mathcal{U} \quad \text{and} \quad m_+ \in \mathcal{C}(L^2)
\]
Now

\[ k(\alpha) = \frac{1}{2} \int_{\mathbb{R}} e^{i\alpha y} \int_{\mathbb{R}} e^{-x^2} \, dx \, \, dy. \]

\[ = \frac{1}{2} \int_{\mathbb{R}} \frac{\alpha y}{\alpha^2 + y^2} \left( \int_{\mathbb{R}} e^{i(x-y)\alpha} \, dx + \int_{\mathbb{R}} e^{i(x+y)\alpha} \, dx \right) \]

\[ = \frac{1}{2} \int_{\mathbb{R}} \frac{\alpha y}{\alpha^2 + y^2} \left( -\frac{1}{i\alpha - \alpha} + \frac{1}{i\alpha + \alpha} \right) \]

\[ = \frac{1}{2} \int_{\mathbb{R}} \frac{\alpha y}{\alpha^2 + y^2} \left( \frac{-i\alpha - y + i\alpha - y}{(\alpha^2 + y^2)} \right) \]

\[ = \int_{\mathbb{R}} \frac{\alpha y}{\alpha^2 + y^2} \, dy \]

\[ = \int_{\mathbb{R}} \frac{du}{1 + u^2}. \]

Now observe that \( k(1) = 1 \) and \( k(0) \)

\( v(3) = 1 - k(3) = 0 \) as \( 3 = 1 \). This means that

\( \text{IRIR} \) above is singular. To remedy the problem we must introduce more smoothness on \( u \) or equivalently we must \( f(x) \) decay faster in \( L^2(\mathbb{R}+) \) in order enough to contain our solution.
To see how to do this, let $0 < p < 1$

and set

\[(14.1)\]
\[\tilde{f}_e (x) = f(x) e^{-p x}\]

then \[(14.2)\] takes the form

\[(14.2)\]
\[\tilde{f}(x) = \int_0^\infty \tilde{h}(x-y) \tilde{f}(y) \, dy\]

where
\[\tilde{h}_e (x) = e^{-p x} \tilde{h}(x)\]

and we have preserved the Wiener-Hopf form of the equation, and also, from \[(14.1)\] we see that
\[\tilde{f}_e \in L'(\mathbb{R})\] We must now show that

We have

\[(14.3)\]
\[\tilde{\omega}_+ = \tilde{\omega}_- \tilde{\omega}(1)\]

has a non-trivial solution \(\tilde{\omega}_e \in DC(L^1)\)

where \(\tilde{\omega}(1) = (1 - \tilde{E}(1))^{-1}\), provided \(|E(1)| > 1\)
\[ N(w) = \frac{i}{2} \int_{0}^{\infty} e^{ iz \cdot x} e^{-px} e^{-1x \cdot jy} \, dy \]

\[ = \int_{0}^{\infty} \frac{d \mu}{1 + u^2(3 + ip)^2} \]

and so for \( z \in \mathbb{C} \)

\[ 1 - N(w) = \int_{0}^{\infty} \frac{u^2 (3 + ip)^2}{1 + u^2(3 + ip)^2} \, du \]

\[ = \int_{0}^{\infty} \frac{u^2 (3 + ip)^2 (1 + u^2(3 - ip)^2)}{1 + u^2(3 + ip)^2} \, du \]

\[ = \int_{0}^{\infty} \frac{u^2 \left[ (3 + ip)^2 + u^2 (3^2 + p^2)^2 \right]}{1 + u^2(3 + ip)^2} \, du \]

\[ = \int_{0}^{\infty} \frac{u^2 \left[ (3^2 - p^2 + u^2(3^2 + p^2)^2) + 2ip \, \xi \right]}{1 + u^2(3 + ip)^2} \, du \]

\[ (1.5.1) \quad 1 - N(w) = \int_{0}^{\infty} \frac{u^2 (3^2 - p^2 + u^2(3^2 + p^2)^2)}{1 + u^2(3 + ip)^2} \, du \]

\[ + 2ip \, \xi \int_{0}^{\infty} \frac{u^2 \, du}{1 + u^2(3 + ip)^2} \]
Note that
\[1 + u^2(z + ip)^2 = 0 \Rightarrow z + ip = \frac{\pm i}{u}.
\]
\[e^{i \varphi} z = i \left( \frac{\pm i}{u} - p \right).
\]
As \(1/u \geq 1\) and \(p < 1\), we see that
\[1 + u^2(z + ip)^2 \neq 0 \quad \text{for} \quad z \in \mathbb{R}, \quad u \in [0, 1].
\]
Thus \(\tilde{c}(z)\) has no singularities on \(\mathbb{R}\); that is of course obvious a priori, as \(\tilde{c} \in C^1(\mathbb{R})\).

From (145.1) we see that \(1 - \tilde{c}(z) = 0\)

in particular that \(\Re \{1 - \tilde{c}(z)\} = 0\) and \(\Re \{1 - \tilde{c}(z)\} = 0\).

But if \(z = 0\),
\[
\Re \{1 - \tilde{c}(z)\} = \int_0^1 u^2 \frac{(-p^2 + u^2 p^2)}{1 + u^2 (ip)^2} \leq 0
\]
\[
= -p^2 \int_0^1 u^2 \frac{(1 - p^2 u^2)}{1 - u^2 p^2} < 0
\]
Thus
\[1 - \tilde{c}(z) = 0\]
and no \(\tilde{c}(z)\) is non-singular.
We now compute the winding number of \( 1 - \tilde{c}(z) \) as \( z \) goes from \(-\infty\) to \(+\infty\).

As \( z \to \pm \infty \), \( 1 - \tilde{c}(z) \to 1 \)

but for \( z < 0 \), we see that \( \Re \left[ 1 - \tilde{c}(z) \right] < 0 \)

and as \( z \) increases from \(-\infty\), \( 1 - \tilde{c}(z) \) stays

in \( \mathbb{C} \) until \( z = 0 \). At \( z = 0 \), as shown above,

\[ \Re \left( 1 - \tilde{c}(0) \right) < 0 \]

And then for \( z > 0 \), \( \Re \left[ 1 - \tilde{c}(z) \right] > 0 \). Thus we have

\[ \int_{-\infty}^{\infty} \frac{\sin^2 u}{(1 - u^2) - \rho^2} \, du \]

The trajectory of \( 1 - \tilde{c}(z) \) from \(-\infty\) to \(+\infty\).

Hence the winding number of \( 1 - \tilde{c}(z) \) is \(-1\).
\[ \text{trivial \# of } \mathcal{O}(3) = (1 - \mathfrak{E}(3))^{-1} = +1. \]

By the general theorem 174, this implies

\[ \left( \begin{array}{c} \text{transformed} \\ \text{from } \Sigma = \{ |x| = 1 \} \\ \text{to } \Sigma = \mathbb{R} \end{array} \right) \]

that

\[ \dim \ker (I - \mathfrak{E}) = 1 \]

and so there is a 1-dimensional family of solutions

\[ \tilde{u}_{+} = \tilde{u} \_ \mathfrak{E}, \quad \tilde{u} \_ \in \mathcal{O}(L^2) \]

This then shows that the failure equation

when \( u \) is solution of \( \mathfrak{F} + \mathfrak{F} e^{-p x} \in L^2(\mathbb{R}^+) \).

A priori \( u \) depends on \( p \) but if \( 0 < p_1 < p_2 < 1 \)

and \( e^{p_1 x} \) and \( e^{-p_2 x} \) are the solutions

obtained by the above procedure, then clearly
To compute $\widetilde{m}_+$ we proceed as follows: set $v^# = \frac{z+ic}{z-ic}$. As winding # $\frac{z+ic}{3-ic} = -1$, this implies winding # of $v^# = 0$. Thus $\log v^#(z) \to 0$ as $z \to \pm \infty$ and we can factorize $v^#$ as $m^# = m_+ v^#$, $m^# \to 1$ as $|z| \to \infty$ using (38).

Explicit formula

$$m^# = e^{\frac{1}{2\pi i} \int_{\gamma} \log v^#(s) \frac{ds}{s-\xi}}$$

Now

(199.4.1) $\widetilde{m}_+ = \frac{\widetilde{m}_- \frac{z-ic}{z+ic} v^#}{3+ic} = \frac{\widetilde{m}_-}{m^#} \frac{z-ic}{z+ic} m^#$

Set

$$h(z) = \frac{\widetilde{m}(z)}{m^#(z)} (\frac{z+ic}{3+ic}), \quad \Re z > 0$$

$$= \frac{\widetilde{m}(z)}{m^#(z)} (\frac{3-ic}{z-ic}), \quad \Re z < 0.$$ By (199.4.1), $h(z)$ is continuous across $\mathbb{R}$ and hence $h(z)$ is entire. As $\widetilde{m}(z) \to 0$ as $z \to \infty$, we see that $h(z) = o(z^\epsilon)$ as $z \to \infty$. Thus $1 = h(\infty)$.
\( (184.1) \)
\[ \mathcal{W}_J(z) = e^{\frac{n_i + i\phi}{3}} \quad \text{for} \quad \phi \neq \pi \]

which is
\[ \mathcal{W}(x) = e^{i\lambda x} \int_{-\infty}^{\infty} \frac{e^{iisx} \mathcal{W}_i(s)}{2\pi i} ds, \quad \lambda > 0 \]

To compute \( \mathcal{W}(0, \psi) \), for \( \psi < \pi < 2\pi \), we use \( 190.1 \).

\[ \mathcal{W}(0, \psi) = \frac{\sec \psi}{\sec \pi} \int_{-\infty}^{\infty} \mathcal{W}(u) du \]

\[ = -\sec \psi \int_{\infty}^{\infty} e^{i\lambda(u + \phi)} \mathcal{W}(u) du \]

\[ = -\sec \psi \int_{\infty}^{\infty} e^{i\lambda(u + \phi)} \mathcal{W}(u) du \]

\[ = -\sec \psi \int_{\infty}^{\infty} \mathcal{W}(u) \left( -i (\sec u + \phi) \right) du \]

\[ = -\sec \psi \int_{\infty}^{\infty} \mathcal{W}(u) \left( -i (\sec u + \phi) \right) du \]

\[ = e^{i\lambda \sec \psi} \int_{-\infty}^{\infty} \log \frac{\mathcal{W}(s)}{s} ds \]

\[ = \frac{e^{i\lambda \sec \psi}}{2\pi i} \int_{\gamma} \frac{ds}{s - (-i (\sec u + \phi))} \]

\[ \left( \lambda + i \right) \]

where \( \mathcal{W}(s) = \frac{\text{strip}}{s - i\phi} \frac{1}{\mathcal{W}(s - i\phi)} \int_{0}^{\text{strip}} \frac{u^2 du}{\lambda^2 + u^2} \]

Thus
\[ \mathcal{W}(0, \psi) = e^{i\lambda \sec \psi} \int_{\gamma} \frac{ds}{s + i\phi + i\sec \psi} \]

(Why is this independent of \( \phi \)?) (\( 184.1 \) made by Mark Mark Mark)
\[ p_1 e^{-p_1 x} = p_1 e^{-p_1 x} e^{-(p_2 - p_1) x} \] solves the same equation as \( p_2 e^{-p_2 x} \) as \( p_1 e^{-p_2 x} \in L^2(0,2) \).

As \( \tilde{\nu}_+ = \tilde{\nu}_- - \tilde{\nu}_+ \in D(L^1) \), has a 1-parameter family of solutions for \( p_2 \), it follows that \( p_1 e^{-p_1 x} = p_2 e^{-p_2 x} \) if \( p_1 = p_2 \).

For more information on the problem, see Dynin + McKean and the references therein.

We now begin our analysis of the asymptotics of some RHP's.

First we consider the Szegö Strong Limit Theorem (see lecture 6).

For \( \varphi \in L^1(\Sigma) \), \( \Sigma = \{ |\beta| = 1 \} \), let

\[ D_n = \text{det} T_n = \text{det} (e_{ij} - k_{i}) \quad 0 \leq i, k \leq n \]
Exercise: (Dyn. + HIC. p.178 et seq.). A model problem for Hilme's equation:

Consider the equation

$$(\text{a.a.t.}) \quad f(x) = \frac{1}{2} \int_0^\infty e^{-(x-y)} f(y) \, dy, \quad x > 0$$

By differentiating twice show that $f(x) = e(1 + x)$ are the only solutions of (a.a.t.1).

Obtain these solutions by utilizing the Wiener-Hopf technique as in Hilme's equation above.
be the associated Toeplitz determinant, where

$$
q_m = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{-im\theta} q(e^{i\theta}) d\theta, \quad m \in \mathbb{Z}
$$

Then if \( q = e^{L(\zeta)} + L'(\zeta) \) and \( \sum_{k=1}^{\infty} k |L_k|^2 < \infty \),

then as \( n \to \infty \)

$$
D_n = e^{(n+1)L_0} + \sum_{k=1}^{\infty} k |L_k|^2 (1 + O(1))
$$

where \( L_m = \int_{\Sigma} e^{-im\theta} L(e^{i\theta}) d\theta \), \( m \in \mathbb{Z} \).

Szego proved this result originally with much stronger assumptions on \( q \), We will prove the theorem under the condition that

1. \( q(\zeta) > 0 \) on \( \Sigma \)

2. \( q(\zeta) \) is analytic in an annulus

$$
|\zeta| < \frac{1}{\rho} < \frac{1}{r} \leq \Sigma
$$

for some \( 0 < \rho < r < 1 \).

Our purpose is to illustrate the steepest descent method.
in one of the simplest RH situations.

Recall from (91.2) (see also P. Leht "Integrable Operators" Ann. Inst. Trans. (2) 1989, 1999 (69-84).)

we have the formula,

\[
\log D_n = - \sum_z \left[ \frac{2}{\pi} \int \frac{F_{e, j}^{(2)}}{G_{e, j}^{(2)}} \, dt \right] \frac{dt}{t}.
\]

Let

\[
\begin{align*}
(F_{e, t}) &= (M_{t+})^{-1} \\
(G_{e, t}) &= (M_{e, t})^{-1} - \frac{3^{-(n+1)}(1 - \varphi_t)}{2\pi i}.
\end{align*}
\]

\[
\varphi_t = (1 - t) + t \varphi(3), \quad 3 \in \Sigma, \quad 0 < t < 1.
\]

and 
\(M_{e, t}\) solve the normalized RHP \((\Sigma, \nu_t)\)

\[
\nu_t = \begin{pmatrix} \nu_0 & - (\varphi_t - 1) 3^{n+1} \\ 3^{-(n+1)}(\varphi_t - 1) & 2 - \varphi_t \end{pmatrix}
\]
The idea of the proof is to move the $z^{n+1}$ term into $|z| < 1$ and the $z^{-n-1}$ term into $|z| > 1$. Then as $n \to \infty$, these terms are exponentially small.

But first we must separate these terms algebraically. This is done using the lower-upper pt.wise factorization of $v_e$:

$$v_e = \left( \begin{array}{cc} 1 & 0 \\ 3^{-n-1} & 1 \end{array} \right) \left( \begin{array}{cc} \psi_e & 0 \\ 0 & \psi_e^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & -\left(1-\psi_e^{1-} \right) 3^{n+1} \\ 0 & 1 \end{array} \right)$$

which is easily verified.

We now utilize the analyticity of $\psi(e)$ in $|e| < |e^{-1}|$. As $|e| > 0$ on $\Sigma$, we see

$$\psi_e = (1 - t) + t \psi(e) \geq c > 0 \quad \text{where} \quad c = \min (1, \psi(e) > 0)$$

Hence $T_{e, \rho} < e_{\rho} \rho^{(n+1)} < 1$ such that
\( q_t'(3) \) is invertible (and analytic) in a neighborhood of 
\[
\rho < \rho^{(1)} < |3| < \rho^{(1)} - | \rho^{(1)} - \rho^{(1)} | < \rho^{(1)} - | \rho^{(1)} - \rho^{(1)} |
\]

Extend \( \Sigma \) to a union of 3 circles,
\[
\Sigma^{(1)} = \{ |3| = \rho^{(1)} \} \cup \Sigma \cup \{ |3| = \rho^{(1)} - | \rho^{(1)} - \rho^{(1)} | \}
\]
all oriented counterclockwise. Define \( m^{(1)} \) as a piecewise analytic function as follows:

\[
m^{(1)}(3) = m_6
\]

\[
m^{(1)}(1) = m_t \left( \frac{1}{1 - (1 - q_t^{-1}) 3^{n+1}} \right)
\]

\[
m^{(1)}(1) = m_t \left( \frac{1}{(1 - q_t^{-1}) i} \right)
\]
A straightforward calculation shows that $\psi^{(1)}$ solves the normalized RHP $(\Sigma^{(1)}, \nu^{(1)})$ when

$$\psi^{(1)} = \begin{pmatrix} 1 & 0 \\ \prod_{j=1}^{n} (1 - (q^{-1})^{-j}) & 1 \end{pmatrix}$$

The RHP's $(\Sigma, \nu) = (\Sigma^{(1)}, \nu^{(1)})$ are clearly equivalent: the solution of one can be obtained from the other by simple algebraic operations,
Now observe that as \( n \to \infty \), \( v^{(1)}(3) = I \)

\( \text{(and exponentially)} \)

\( \text{goes to 0 uniformly for } \beta = \beta^1 = p^{(1)} \beta, ~ |\beta| = |p^{(1)}| \beta \)

\( \text{(and also uniformly for } t \in [0, 17] \). \n
In other words

\( v^{(1)}(3) \) converges uniformly on \( \Sigma^{(1)} \) to the jump matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & \nu_t^{-1}
\end{pmatrix}
\]

on \( \Sigma \)

It then follows that if

\[
(1 - C_{v^{(1)}}) M^{(1)} = I
\]

\[
(1 - C_{v^{(\infty)}}) M^{(\infty)} = I
\]

are the respective associated integral operators

\[
\nu_t M^{(1)} = \frac{1}{1 - C_{v^{(1)}}} I = \frac{1}{1 - C_{v^{(\infty)}}} \left( \frac{1}{1 - C_{v^{(1)}}} - C_{v^{(\infty)}} \right) I
\]
\[ \mu^{(1)} = \frac{1}{1 - (\nu^{(1)} - \nu^{\infty})} \mu^{\infty} + O(e^{-\gamma n}, \text{ some } \gamma > 0). \]

\[ (206.1) \quad \| \mu^{(1)} - \mu^{\infty} \|_{L^1(\Sigma^{(1)})} = O(e^{-\gamma n}) \quad \text{as } n \to \infty. \]

It is an important exercise that in fact

\[ (206.2) \quad \| \mu^{(1)} - \mu^{\infty} \|_{L^1(\Sigma^{(1)})} = O(e^{-\gamma n}) \]

The solution of the normalizer OHP

\[ (\Sigma^{(1)}, \nu^{\infty}) \equiv (\Sigma, \nu^{\infty}) \quad \text{is given by} \]

\[ M^{\infty} = \begin{pmatrix} e^{\gamma t} & 0 \\ 0 & e^{-\gamma t} \end{pmatrix} \]

When \[ \gamma t = \int_{\Sigma} \log \frac{\theta_t(s)}{s} \frac{ds}{2\pi i} \]

It follows in particular from (206.2) that on \( \Sigma \)

\[ M^{(1)}(\Sigma) - M^{(\infty)}(\Sigma) \]
uniformly for \( z \in \Sigma \). Moreover (exercise) the derivatives also converge
\[
\frac{d}{dz} m_{2n+1}(z) \sim \frac{d}{dz} m_{2n}(z), \quad n \to \infty
\]
uniformly for \( z \in \Sigma \).

We see in particular from the relations between \( m_{t+1}(z) \) and \( m_t(z) \) in
\[
\begin{pmatrix}
1 & -(1-\psi_t^{-1})z^{n+1} \\
0 & 1
\end{pmatrix}
\]
that
\[
m_{t+1}(z) \sim m_t(z) \begin{pmatrix}
1 & -(1-\psi_t^{-1})z^{n+1} \\
0 & 1
\end{pmatrix}
\]
uniformly for \( z \in \Sigma, \quad 0 \leq t \leq 1 \).

Inserting this asymptotic formula into (20.1) (20.2) (20.3) we find (exercise).
\[ \log D_n \approx -1 \int_0^1 \int \sum (1 - \psi(\beta)) \left( 2 \psi_1 q + \frac{n+1}{2} \psi^{-1} \right) \frac{d\beta}{2\pi} \]

Now from
\[ q_{t+} = q_t + (\log \varphi_t) \]
and hence
\[ 2 \psi_1 q_{t+} - q_t \psi_1 q_t^2 = \psi_t (q_{t+} + q_{t-}) \]
we draw
\[ \log D_n \approx - \int_0^1 \int \sum (1 - \psi(\beta)) \left( \psi^{-1}(q_{t+} + q_{t-}) + \frac{n+1}{2} \psi^{-1} \right) \frac{d\beta}{2\pi} \]
\[ = (n+1) L(\psi) + \int_0^1 \int \sum (\psi - 1) \psi^{-1}(q_{t+} + q_{t-}) \frac{d\beta}{2\pi} \]
\[ = \int_0^1 \int \frac{1}{1 + \psi_1} \frac{d\beta}{2\pi} \]
\[ = - \int_0^1 \int \frac{d\alpha}{2\pi} \left( \psi(1 + \psi_1) \frac{d\alpha}{2\pi} \right) \]
\[ = - L(\psi) \]
But for \( \varepsilon > 0 \) small we have the elementary integration identity,

\[
\int \frac{\log \varphi_\varepsilon(s)}{(s-s_\varepsilon)^2} \, ds = \int \frac{\log \varphi_\varepsilon(s)}{(s-s_\varepsilon)^2} \, ds.
\]

\[
\int_{|s|=1} \left( \frac{\psi(s) - 1}{\psi(s)} \right) \, ds = \int_{|s|=1} \left( \frac{\log \varphi_\varepsilon(s)}{(s-s_\varepsilon)^2} \right) \, ds.
\]

Letting \( \varepsilon \to 0 \), we obtain that

\[
\int \frac{\log \varphi_\varepsilon(s)}{(s-s_\varepsilon)^2} \, ds = \int \frac{\log \varphi_\varepsilon(s)}{(s-s_\varepsilon)^2} \, ds.
\]

This proves the strong Szegő limit theorem for analytic \( \varphi(s) \).