Lecture 12

The following result establishes the general relationship between the operators $1 - Cw$ and matrix factorizations.

For definiteness, let $\Sigma$ be a simple, bounded, rectifiable, $A$-regular closed contour in $\mathbb{C}$ with $z = 0$ in its interior.

Let $D(\beta)$ be the diagonal matrix

\[
(173.0) \quad (D(\beta) = \text{diag}(z^{h_1}, z^{h_2}, \ldots, z^{h_m})
\]

with $h_1 > h_2 \geq \ldots \geq h_m$, integers.

Let $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$. We say that $v, v^{-1} \in L^\infty(\Sigma)$ has a generalized (right-)standard factorization relative to $L^p(\Sigma)$ in case

\[
(173.1(i)) \quad v = m_{-1}^{-1} \Theta(\beta) m_+ \quad \text{on } \Sigma
\]

where

$$m_+ \in A_+ + \text{OC}(L_p)$$

$$m_{-1}^{-1} \in B_+ + \text{OC}(L_d)$$

and $\det A_+, \det B_+ \neq 0$.

(173.1)(ii) The operator $X(z) = (C^+ m_{-1}^{-1}) m_+ \in L(L^p(\Sigma))$. 

\[173\]
The integers $b_1, \ldots, b_n$ are called the (right-) partial indices of $\nu$.

**Remark:** A similar defn. of generalized (left-) standard factorization, $\nu = m \cdot \tilde{D} \cdot m^-$, etc., can of course be made, but by convention, by a gen. fact. we always mean a (right-) factorization.

The main theorem is the following:

**Theorem 17.9.**

Let $\Sigma$, $\nu$ be as above and $1 < p < \infty$. In order that the matrix function $\nu$ admits a generalized factorization relative to $L^p(\Sigma)$, it is necessary and sufficient that the operator $1 - \nu = I - C^-(\nu - I)$ is Fredholm in $L^p(\Sigma)$. 
(75.1) Some properties of a factorization: Let \((m_+, m_-, D)\) be a gen. stand. factorization for \(\nu\). Then

(i) The partial indices \(k_1, \ldots, k_n\) are uniquely determined (the factors \(m_\pm\) are not)

(ii) \(\text{index } (1 - \nu) = \sum \frac{n}{k_i}\)

(iii) \(\text{dim } \ker (1 - \nu) = \sum_{k_i > 0} k_i\)
\(\text{dim } \text{coker } (1 - \nu) = \sum_{k_i < 0} |k_i|\)

(iv) In general the \(k_i\)'s are not stable under perturbations \(\nu \to \nu + \delta \nu\). Only \(\sum k_i = \text{index } (1 - \nu)\) is stable.
However, if \(0 < k_i - k_{i+1} < 1\), \(i = 1, \ldots, n-1\), then the \(k_i\)'s are stable.

(v) The indices \(k_i\) depend in general on \(p\). Moreover \(\text{ind } (1 - \nu)\) can also depend on \(p\).

(vi) Theorem 157.1 is a special case of Theorem 174.1 with \(B(x) = I\).

(vii) If \(\nu = \nu^- \nu^+ = (I - \nu^-) (\nu^+ + \nu^-)\), \(\nu^+, \nu^- \in C^0(\delta)\) is a pt. wise factorization of \(\nu\), then (is Fredholm)

(a) \(1 - (\nu + \nu^-)\) is Fredholm \((=)\)
\(1 - (\nu^-) = 1 - (\nu^+ + \nu^-)\)
(b) \( \dim \ker (1 - \omega) = \dim \ker (1 - \omega) \)
\( \dim \text{coker} (1 - \omega) = \dim \text{coker} (1 - \omega) \)

so that if \( (1 - \omega) \) is dense \( (1 - \omega) \) is Fredholm, then

\[ \text{ind} (1 - \omega) = \text{ind} (1 - \omega) \]

(Viii) condition (1.73.1) (ii) cannot be omitted.

We will prove some of these facts; the remaining proofs can be found in Clancy-Phillips or Spitkovsky-Litvinchuk.

(c1) Suppose \( \omega \) has 2 gen. str. factorizations

\[ m_+^{-1} D m_+ = \tilde{m}_-^{-1} \tilde{D} \tilde{m}_+ \]
\[ D = \text{diag} (\tilde{\delta}_1, \ldots, \tilde{\delta}_n) \quad \tilde{D} = \text{diag} (\tilde{\delta}_1, \ldots, \tilde{\delta}_n) \]
\[ b_1 \geq b_2 \geq \ldots \geq b_n \]
\[ b_1 > b_2 \geq \ldots \geq b_n \]

(1.76.1) Then

\[ C - D = \tilde{D} C_+ \]

where \( C = \tilde{m}_- m_+^{-1} \), \( C_+ = \tilde{m}_+ m_+^{-1} \)

by a familiar argument \( C_+ = E_+ + C_+ h \)

where \( \det E_+ \neq 0 \), and \( h \in L^p + L^q \).
From (176.1) 

(177.1) \quad (C^-)_{ij} = (C^+)_{ij} \frac{\hat{k}_i - k_j}{2} \quad (i, j) \in n.

Now assume that for some \( l \), \( 1 \leq l \leq n \),

\[ \frac{\hat{k}_i}{l} > k_e \] (the case \( \frac{k_i - k_e}{l} \) is similar — exercise).

Then \( \frac{\hat{k}_i - k_i}{l} > 0 \) for \( 1 \leq i \leq l, \ l \geq i \leq n \).

It follows that from (177.1) that for \( 1 \leq i \leq l, \ l \geq i \leq n \),

\[ h_{ij} (3) = (c^-)_{ij} (3) \quad \text{3 outside } \Sigma \]

\[ \begin{array}{c}
\uparrow \\
\text{continua of } C^- \text{ outside } \Sigma
\end{array} \]

\[ = (C^+)_{ij} (4) \frac{\hat{k}_i - k_j}{2} \quad \text{2 inside } \Sigma \]

\[ \begin{array}{c}
\uparrow \\
\text{continua of } C^+ \text{ inside } \Sigma
\end{array} \]

is entire and bounded in \( C \). Hence \( h_{ij} (3) = \text{const}_{i,j} \).

But as \( \frac{\hat{k}_i - k_j}{l} > 0 \), \( \text{const}_{i,j} \) must \( = 0 \).

It follows that \( (C^-)_{ij} = 0 \) and no \( C^- \) has

the block form

\[ C^- = \delta \begin{pmatrix}
\begin{array}{c|c}
\begin{array}{c}
X_{l-1} \end{array} & n-l+1 \\
\hline
0 & X
\end{array}
\end{pmatrix}
\end{pmatrix} \]
from which we see that rank $C_\neq \leq n-1 + m-n = n-1$.

This is a contradiction as $C^+$ has range on $\Sigma$. Thus

$\Theta^{(3)} = \tilde{\Theta}^{(3)}$.

(iii) (iii)(iv), see refs.

(v) Consider the following example on $\Sigma = \{ x_1 = 1 \}$

$$v(y) = 1, \quad 3 \in \Sigma, \quad 3 \in C^+$$

$$= -1, \quad 3 \in \Sigma, \quad 3 \in C^-$$

Seek

$$\phi^n m_+ = m_- v^-$$

and $m_+ \in \mathcal{H} + O(1)$

Let $h(y) = \left( \frac{3 y-1}{3+y} \right)^2$ which is analytic in

$$\mathbb{C} \backslash [0,1]$$

and $h(y) \to 1$ as $y \to \infty$.

Then clearly $h_+(y) = h_-(y)$ on $\Sigma$, $3 \in C^+$.

and

$h_+ = -h_-$ on $\Sigma$, $3 \in C^-$

Thus

$v = h^- h_+$. on $\Sigma$.

Hence $\Theta^{(3)} = \phi^n m_+ h^- = m_- h_-$.
Assume first that \[ p > 2. \]

By familiar arguments \( l_{(3)} \) is analytic in \( C \setminus \{1 \sim 0 \sim 15 \} \)

But \( h_+^{-1} = 1 + O(c \ell_3) \) for any \( 1 < s < 2 \) and

so \( m_\pm h_+^{-1} \leq 1 + C(c \ell_3) \) where \( 1 = \ell_1 + \ell_2 + \ell_3 \)

and \( \frac{1}{s} = \frac{1}{p} + \frac{1}{s} < 1 \), provided we choose, as we can,

so sufficiently close to 2. Thus for \( p > 2 \), \( l_{(3)} \) is

analytic across \( \Sigma \) and bounded as \( \ell \to \infty \).

Now if \( n > 0 \). We see that \( l_{(3)} \) is analytic in \( C \),

and as it is bounded at \( \infty \), it must be constant. But then

\( n > 0 \implies c \text{ constant} = 0 \implies m_\pm = 0 \). Contradiction!

On the other hand if \( n = 0 \), then \( l_{(3)} = c \text{ constant} = 1 \).

So no \( m_\pm = h_\pm \) But \( h_\pm \leq 1 + O(c \ell_3) \)

for \( p > 2 \). Again a contradiction.

So suppose \( n = -\hat{n} < 0 \).
\[ l(n) \to m = m + h \to m - h \text{ and so} \]

\[ l(1) \to m = p(1) = m + - \text{ is a monic polynomial} \]

Thus
\[
\begin{align*}
m_+ &= p(3) \, h^2 - p(3) \left( \frac{2}{3+1} \right)^2 \\
m_- &= p(3) \, h^2 - p(3) \left( \frac{2}{3+1} \right)^2 \, \hat{n}
\end{align*}
\]

Now as \( m = \) are invertible, as their extensions to \( \{ \mathbb{F}(1) \}, \{ \mathbb{F}(1) \} \) resp. are invertible, it follows that \( p(1) \) can only have zeroes at \( b = \pm 1 \). If \( p \) had a zero at \( b = 1 \), then as \( b \to 1 \) \( m_1(b) \to (3-1)^2 - m \)

where \( m > 1 \). and so \( m_+ \to (3-1)^{2+m} \) \text{ for any} \( 1 < d < 2 \), \( \frac{1}{q} + \frac{1}{p} = 1 \). Hence \( p(1) \) cannot have a zero at \( b = 1 \). and so \( p(1) = (3+1)^n \). Thus

\[ \hat{n} \to (3+1)^{1/2} \text{ which lie in } \mathbb{Q} \]

\[ 1 < d < 2 \text{ if and only if } \hat{n} = 1. \]

Thus \( m_+^{1/3-1} m_+ = 5 \text{ in a factorization} \)
\[ m_+ = \frac{3+1}{3+1} \left( \frac{3-1}{3+1} \right)^{\frac{1}{2}} \]
\[ m_- = \frac{3+1}{4} \cdot \left( \frac{3-1}{3+1} \right)^{\frac{1}{2}} \]
\[ m_+ \in 1 + \mathcal{O}(n) , \quad (m_-)^{-1} \in 1 + \mathcal{O}(n) \]

Furthermore, it is an important exercise to show that under $X = (C \cdot m_+)^{-1} m_-$

\[ -L^p \] . Thus, $(m_+, D = \frac{1}{2})$ is a general fact.

For $v$ with index $= -1$.

**(81.1) Exercise (a)** Use the above methods to show that index $\neq -1$ for $1 < p < 2$

**(b)** Show that $1 - v$ is not Fredholm for $p = 2$.

Prove this by showing no fact $2^m m_+ = m_-$ with $m_+, m_- \in 1 + \mathcal{O}(n)$ . Alternatively, show that both columns $(1 - v) = 0 \forall p > 2$.

**(HINTS: Consider Jordan block $M = M - M + F$ with rational $F$)**

**(vi)**

**(vii)** For $v = v_- v_+ = (I - w^-)^{-1} (I + w^+)$

\[ (1 - C_w) h = h - C^{-1} h w_+ - C^{-1} h w_- \]
\[ = h - C^{-1} h w_+ - C^{-1} h w_- - h w_- \]
\[ = h v_- - C^{-1} h (w_+ + w_-) = h v_- - C^{-1} h (v_+ - v_-) \]
\[ = h v_- - C^{-1} h v_+ (v_-^{-1} u_+ - 1) = (1 - C^{-1} : v_-^{-1} ) h v_+ \]
\[(182.1) \quad (1 - (\omega) = (1 - (\nu) R_{\nu})\]

where $R_{\nu}$ is right multi. by $\nu$.

(\text{Vii1 (a,b)}) now follow immediately from (182.1)

\[\nu = m^{-1}m^+ \quad \text{where} \quad m^+ \in I + O(L^p), \quad m^{-1} \in I + O(L^q)\]

but $X = (C^+, m^{-1}, m^+ \in L^p(\Sigma))$ can be constructed.

\text{Using same calculations in Spitkovsky - Litvinchuk pp 114-116.}

In these pages the authors construct a function $k(\theta) > 0$ on the unit circle $\Sigma$ with the following properties:

- $k \in L^p(\Sigma)$ \quad $1/p < \infty$ \quad and $k^{-1} \in L^\infty(\Sigma)$.

For the intervals $\Theta_s = \{ e^{i\theta} : 2^{-1 - 2^{-s - 1}} < \Omega < 2^{-2^{-s - 1}} \}$

\[\text{w} \Sigma, \quad s = 1, 2, 3, \ldots \]

\[(182.2) \quad \frac{1}{18\pi} \left( \int_{\Theta_s} k(\theta)^p d\theta \right)^{1/p} \left( \int_{\Theta_s} k(\theta)^{-q} d\theta \right)^{1/q} = \frac{1}{3} (2+8^p)^{1/p} (2+8^q)^{1/q} \]
which goes to \( 0 \) as \( s \to \infty \).

Now a necessary and sufficient condition for
\[
T = (c^+ b^{-1/2}) h
\]
to be bounded in \( L^p(\Sigma) \) is

\[
\sup_{\Sigma} \frac{1}{|x|} \left( \int_{\Sigma} h^p d\theta \right)^{1/p} \left( \int_{\Sigma} k^{-q} d\theta \right)^{1/q} < \infty
\]

where the sup is taken over all intervals in \( \Sigma \).

It follows from (18.1.2) that \( T \) is not bounded in any \( L^p(\Sigma) \).

Now set
\[
f_1(z) = e^{i\frac{z}{h}} \int_{-\pi}^{\pi} \left( \frac{e^{i\theta}}{\cos \theta - \frac{z}{h}} \right) \log h(\theta) \, d\theta,
\]
\( |z| < 1 \).

Then a direct calculation shows that
\[
|f_1(z)|^2 = k^2 \quad \text{and} \quad |f_1(e^{i\theta})| = h(\theta).
\]

Set
\[
(18.3.1) \quad v(z) = f_1(z) \sqrt{f_1(z)'} = \frac{z}{h^2(z)}, \quad z \in \Sigma
\]
Here \( u \in L^\infty \), \( \psi' \in L^\infty \)

Set \( m_+(\psi) = \frac{\psi(\psi)}{\psi(\psi)} \)

\[ m_-(\psi) = \frac{\overline{\psi(\psi)}}{\overline{\psi(\psi)}} = \frac{\psi(\psi)}{\psi(\psi)} \]

Then \( u = m_- m_+ \) is a factorization of \( u \)

with \( m_+ \in C + \mathcal{O}(L^p) \), \( m_- \in C^{-1} + \mathcal{O}(L^q) \)

for some \( c > 0 \), by the property of \( k \).

However, writing \( m_+ = k e^{i\phi} \), we have for \( h \in L^q(\mathbb{E}) \)

\[ X h = (C^h m_+^{-1}) m_+ = (C^h e^{-i\phi} m_-^{-1}) k e^{i\phi} \]

\[ = R e^{i\phi} (T \overline{R} e^{-i\phi} h) \]

and so \( \overline{X} \) unboundedness of \( T = \) boundedness of \( X \).

Thus \( u = \frac{F}{F} \) has an "\( L^p, L^q \)" factorization:

but not a standard factorization.
Finally we illustrate some of the properties

(i) \ldots (viii) with some $2 \times 2$ examples.

Let $\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

For $a \neq 0$, set $v_a^+(\gamma) = \begin{pmatrix} 3 & a \\ 0 & 3^{-1} \end{pmatrix}$

Direct computation shows that

\[
(185.1) \quad v_a^+ = \begin{pmatrix} 3 & a \\ 0 & 3^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{a^2} & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} = -a^{-1} \begin{pmatrix} -\frac{1}{a} & 0 \\ -\frac{1}{a^2} & 1 \end{pmatrix} \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \quad \overbrace{m_2}^{A(\alpha) = I} \quad \Rightarrow k_1 = k_2 = 0
\]

\[
(185.2) \quad \text{Again for } a \neq 0, \quad \text{now set } v_a^-(\gamma) = \begin{pmatrix} 3 & 0 \\ a & 3^{-1} \end{pmatrix}
\]

\[
\text{Direct computation shows that }
\]

\[
v_a^- = \begin{pmatrix} 3 & 0 \\ a & 3^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{a}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ \frac{1}{a} & 1 \end{pmatrix} \quad \overbrace{m_1}^{A(\alpha) = I} \quad \Rightarrow k_1, k_2 = -1
\]
For $a = 0$

$$v_0 = v_0^k = v_0^l = (3 0 1) = I^{-1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} I$$

Thus we see that the $k_i$ are not stable under perturbation

$$v_0 \to v_0^k, \quad D_0 = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow D_0^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Of course, other perturbations preserve the $k_i$:

$$D_a = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} = D_0$$

In both cases, however, $k_i + k_i$ is preserved.

It's no index is stable.

Now consider for $a \neq 0$

$$V_a k_1 = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow m_-^{-1} \sqrt{D_a k_1} m_+$$

$$V_a l_1 = \begin{pmatrix} 3 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow m_-^{-1} \sqrt{D_a l_1} m_+$$
And now in both cases

\[ v_{\bar{a}}^t = \Phi_{\bar{a}} = (\Phi_{\bar{a}} \circ \Phi_{\bar{a}}) = v_{\bar{a}} \]

when \( V_0 = (\Phi_{\bar{a}} \circ \Phi_{\bar{a}}) = I \), so the partial indices are stable under these perturbations.

The difference between the cases \( v_{\bar{a}}^t, v_{\bar{a}}^t \) and \( v_{\bar{a}}^t, v_{\bar{a}}^t \) is that \( |b_1 - b_2| > 1 \) for \( v_0 \) when,

\[ |b_1 - b_2| = 1 \] for \( V_0 \) (cf. (IV) p. 175).

Exercise: Show that the \( b_i \)'s are indeed stable for all perturbations \( V_0 \to U \) of \( V_0 \).