Oct 4, 2018

Linear Algebra

Problem set #3

Due: Oct 11, 2018

1. Show by Gaussian elimination that the only left nullvectors of

$$M = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 3 \\ 3 & 4 & 6 & 2 \end{pmatrix}$$ are multiples of $\ell = (1, -2, -1, 1)$. Then

use the fact that for a linear map $T$, $R_T^{\perp} = N_{T'}$ to conclude that the condition $0 = u_4 - u_3 - 2u_2 + u_1$ is (necessary and) sufficient to solve the system $Mx = u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$.

2. Suppose $T \in \mathcal{L}(X)$, $\dim X = n$ and let $B : X \to \mathbb{R}^n$ be an isomorphism such that $B\alpha_i = e_i$, $i = 1, \dots, n$, for some basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ for $X$. Let $M = BTB^{-1} \in \mathcal{L}(\mathbb{R}^n)$

and let $M_{ij} = (Me_j)_i$ be the matrix associated with $M$ as in Thm 1, p32 (Lax). Show that $T\alpha_j = \sum_{i=1}^{n} M_{ij} \alpha_i$, $j = 1, \dots, n$. Thus $(M_{ij})$ is the matrix for $T$ in the basis $\mathcal{B}$.

3. Let $S$ be a linear operator in $\mathbb{R}^2$ such that $S^2 = S$ i.e. $S$ is a projection. Show that either $S = 0$ or $S = 1$ or $S\alpha_j = \sum_{i=1}^{2} A_{ij} \alpha_i$, $j = 1, 2$ for some basis $(\alpha_1, \alpha_2)$ for $\mathbb{R}^2$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

4. Let $X$ be an $n$-dimensional vector space over a field $K$, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for $X$.

(a) Show that there is a unique linear operator $T$ on $X$ such that $T\alpha_j = \alpha_{j+1}$, $j = 1, \ldots, n-1$, $T\alpha_n = 0$.
What is the matrix $A$ of $T$ in the basis $\mathcal{B}$, i.e., $T\alpha_j = \sum_{i=1}^{n} A_{ij}\alpha_i$, $j = 1, \ldots, n$.

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let $S$ be any linear operator on $X$ such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is a basis $\mathcal{B}'$ for $X$ such that the matrix for $S$ in the basis $\mathcal{B}'$ is the matrix $A$ in part (a).

(d) Prove that if $M$ and $N$ are $n \times n$ matrices over $K$ such that $M^n = N^n = 0$ but $M^{n-1} \neq 0$, $N^{n-1} \neq 0$, then $M$ and $N$ are similar.

5. Let $W_1$ and $W_2$ be subspaces of a finite-dimensional vector space $X$.

(a) Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

(b) Prove that $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$