1. This problem provides a dynamical proof of the spectral theorem
(Thm 4.1 in lax) for real self-adjoint matrices $M_0$. There exist a diagonal real matrix $D$ and an orthogonal matrix $Q$, $Q^T Q = Q Q^T = I$, such that $M_0 = Q D Q^T$.

**Proof:** Let $M_0 = E_{M_0} - M_0^T$ be a given $n \times n$ matrix.

Consider the differential equation for $M = M(t)$

\[
\frac{dM}{dt} = [M, B(M)] = MB(M) - B(M^T) M
\]

\[M(t=0) = M_0\]

where $B(M) = M_2 - M_2^T$ where $M_2$ is the strictly lower triangular part of $M$. Thus $B = B(M)$ is skew-symmetric, $B = -B^T$, and

\[
(B(tM))_{ij} = M_{ij} \quad \text{if} \quad i > j
\]
\[
= 0 \quad \text{if} \quad i = j
\]
\[
= -M_{ij} \quad \text{if} \quad i < j
\]

Equation (1) is called the **Toda equation**.

(i) Show that, by standard ODE techniques, (1) has a
unique local solution $\alpha(t)$, $0 \leq t < T$, for some $T > 0$, where $\alpha(0) = \alpha$. 

(iii) Show that the constant $\alpha(t)$ is and conclude that (i) has a unique global solution $\alpha(t)$, $0 \leq t < \infty$, $\alpha(0) = \alpha$. 

(iii) Let $M(t)$ be the solution of (1) and let $B = B(M(t))$. Show that the equation

\[
\frac{d\Theta}{dt} = \Theta B
\]

has a unique global solution $\Theta = \Theta(t)$, $t > 0$, $\Theta(0) = I$. 

(iv) Show that $\Theta$ is orthogonal for all $t > 0$. 

(v) Show that $M(t)$ is orthogonal for all $t > 0$.

(vi) Conclude from (vi) that

\[
\frac{d\Theta}{dt} = \Theta B
\]

isospectral.

(vii) Show that

\[
\frac{dM}{dt} = 2 \sum_{j=2}^{n} M_{jj}^2 > 0
\]
and conclude using (3)

\[ \sum_{i,j=1}^{n} M_{ij}(t) = \text{const} \]

That

\[ M_{ii}(0) = \lim_{t \to \infty} M_{ii}(t) \]

exists and

\[ M_{ii}(0) = M_{ii}(0) + \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} M_{ij}^{2}(t) \text{ at} \]

By (11) and (3) it follows that \( \frac{d}{dt} \alpha(t) \) is bounded

for all \( t > 0 \). Conclude from (9) that

\[ \sum_{j=1}^{n} M_{ij}^{2}(t) \to 0 \quad \text{as} \quad t \to \infty \]

(v) Show similarly that \( \frac{d}{dt} (M_{i1} + \ldots + M_{i1}) \to 0 \)

and conclude that for any \( 1 \leq k \leq n \)

\[ \lim_{t \to \infty} M_{kk}(t) = M_{kk}(0) \]

exists and

\[ \sum_{j=1}^{k} \sum_{j=1}^{k} M_{ij}^{2}(t) \to 0 \]

Thus \( \lim_{t \to \infty} M(t) = 0 \) exists with \( D \) diagonal.
Finally, as the orthogonal matrices form a compact set, there exists \( t_n \to \infty \) such that \( Q(t_n) \to Q(\infty) \) for some orthogonal matrix \( Q(\infty) \). But from (6), we have

\[
M(t_n) = Q(t_n)^T M_0 Q(t_n)
\]

and letting \( t_n \to \infty \), we conclude that

\[
D = Q(\infty)^+ M_0 Q(\infty) \tag{13}
\]
or

\[
M_0 = Q(\infty) D Q(\infty)^T
\]

which proves the spectral theorem. Necessarily, all elements of \( D \) are the eigenvalues of \( M_0 \).

2. Give a dynamical proof of the spectral theorem for (complex) self-adjoint matrices \( M = M^* \), i.e., there exists a real diagonal matrix \( D \) and a unitary matrix \( Q \),

\[
Q^* Q = \mathbb{1} = Q Q^*
\]
such that \( M_0 = Q D Q^* \).

(Hint: Consider the differential equation

\[
\frac{dX}{dt} = [X, B(t) X], \quad t > 0, \quad X(0) = M_0.
\]
where \( D(2n) = \mathbf{M} - \mathbf{M}^* \).

3. Use Theorem 4.5. Show in (1) to prove (2).

Wielandt-Hoffman inequality, viz., let \( \mathbf{M} \) and \( \mathbf{N} \) be real, symmetric matrices with eigenvalues \( m_i \) and \( n_j \) arranged in increasing, or decreasing order. That is

\[
(1) \quad \sum (m_i - n_i)^2 \leq \text{tr}(\mathbf{N} - \mathbf{M})^2
\]

Proof:

(i) Show that it is enough to prove that

\[
(2) \quad \sum_{i,j} m_i n_j \geq \text{tr} \mathbf{NN}^T
\]

in the case that

(a) \( (m_i) \) and \( (n_i) \) are in decreasing order

(b) \( \mathbf{M} \) is diagonal, i.e., \( \mathbf{M} = \text{diag}(m_1, \ldots, m_n) \)

d (2) becomes

\[
(3) \quad \sum_{i,j} m_i n_j \geq \sum_{k=1}^{n} m_k n_{kk}
\]

where \( n_{kk} = N_{kk} \).

(ii) Summing by parts, show that

\[
(4) \quad \sum_{k=1}^{n} m_k n_{kk} = m_n \sum_{k=1}^{n} n_{kk} + \sum_{k=1}^{n-1} (m_k - m_{k+1}) \left( \sum_{i=1}^{k} n_{ii} \right)
\]
Now under the Todd flow \( \text{flow}(1) \) of the problem I applied to \( N(t) \),

with \( M \) fixed, \( N(0) = N \),

\[
\sum_{k=1}^{n} \tau_{kk} = \tau N(t) = \text{constant}
\]

and

\[
\sum_{j=1}^{k} \tau_{ij} = \sum_{j=1}^{k} \tau_{ij}(t) \text{ is increasing in time.}
\]

Thus as \( m_k > m_{k+1} \), it follows that \( \sum_{k=1}^{n} m_k \tau_{kk}(t) \)

is non-decreasing. Also as \( t \to \infty \), \( \tau_{kk}(t) \to d_k \)

where \( d_k, k=1, \ldots, n \) are the eigenvalues of \( N \).

Thus

\[
\sum_{k=1}^{n} m_k \tau_{kk}(t)
\]

\[
+ N(t) = \sum_{k=1}^{n} m_k \tau_{kk}(0) = \sum_{k=1}^{n} m_k \tau_{kk}(t) = \sum_{k=1}^{n} m_k d_k
\]

Now the \( d_k \)'s are the eigenvalues of \( N \), but

they may be out of order. However

(iii) Show that if \( m_1 > \cdots > m_n \), and \( \hat{d}_1, \ldots, \hat{d}_n \)

is any other set of numbers, then

\[
\sum_{k=1}^{n} m_k \hat{d}_k \leq \sum_{k=1}^{n} m_k d_k
\]

where \( \hat{d}_1 > \cdots > \hat{d}_n \) is a monotonically decreasing sequence

of \( d_1, \ldots, d_n \). This proves (iii) and hence the Widland-Hoffman
4. Suppose \( T \) is a real tridiagonal matrix

\[
T_{ij} = 0 \quad \text{if} \quad |i-j| > 1.
\]

such that \( T_{i,i+1} \neq 0 \), \( i = 1, \ldots, n-1 \).

(a) Show that each eigenvalue of \( T \) is geometrically simple, i.e., \( \dim \text{Nul} (T - \lambda I) = 1 \) if \( \lambda \) is an eigenvalue of \( T \).

(b) If \( u = (u_1, \ldots, u_n)^T \) is an eigenvector for \( T \), then \( u \neq 0 \).

(c) Show that if \( T \) is real symmetric tridiagonal matrix with non-zero off-diagonal entries, then \( T \) has simple spectrum and if \( u = (u_1, \ldots, u_n)^T \) in an eigenvector, then \( u \neq 0 \).